A Scalable Approach for Automated Precision Analysis

David Boland
and George A. Constantinides

Accelerating an application

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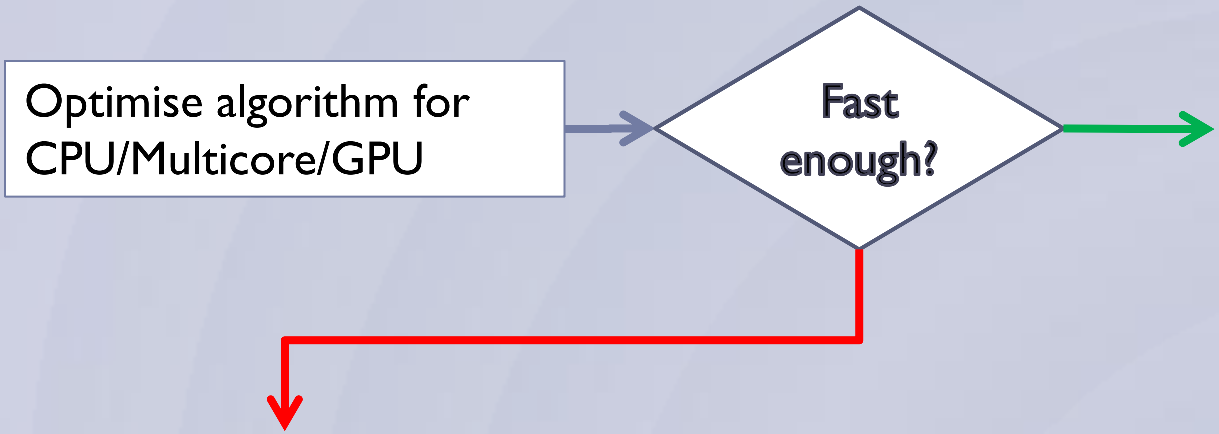

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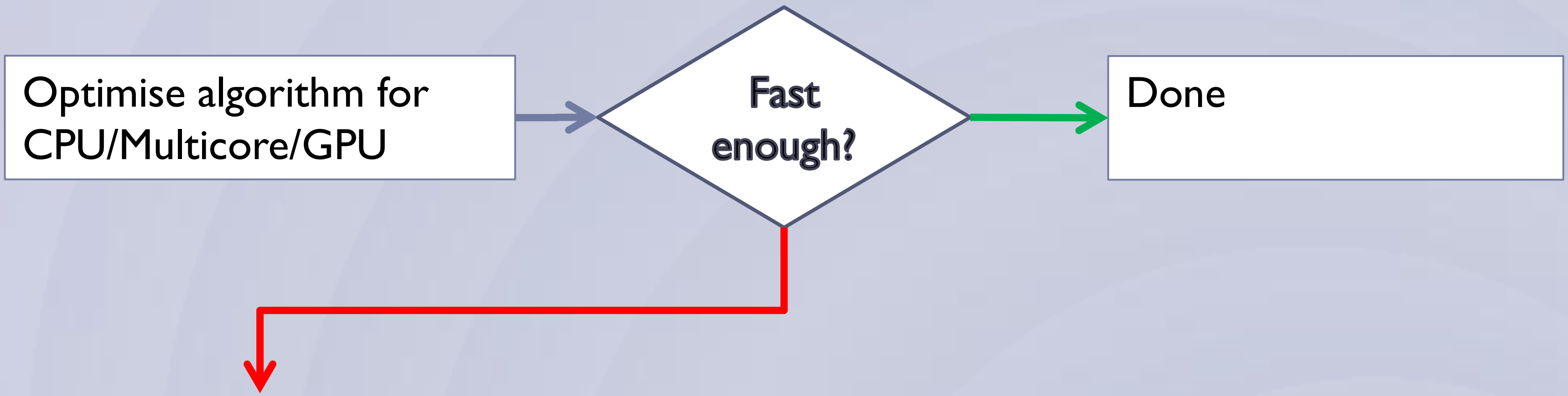

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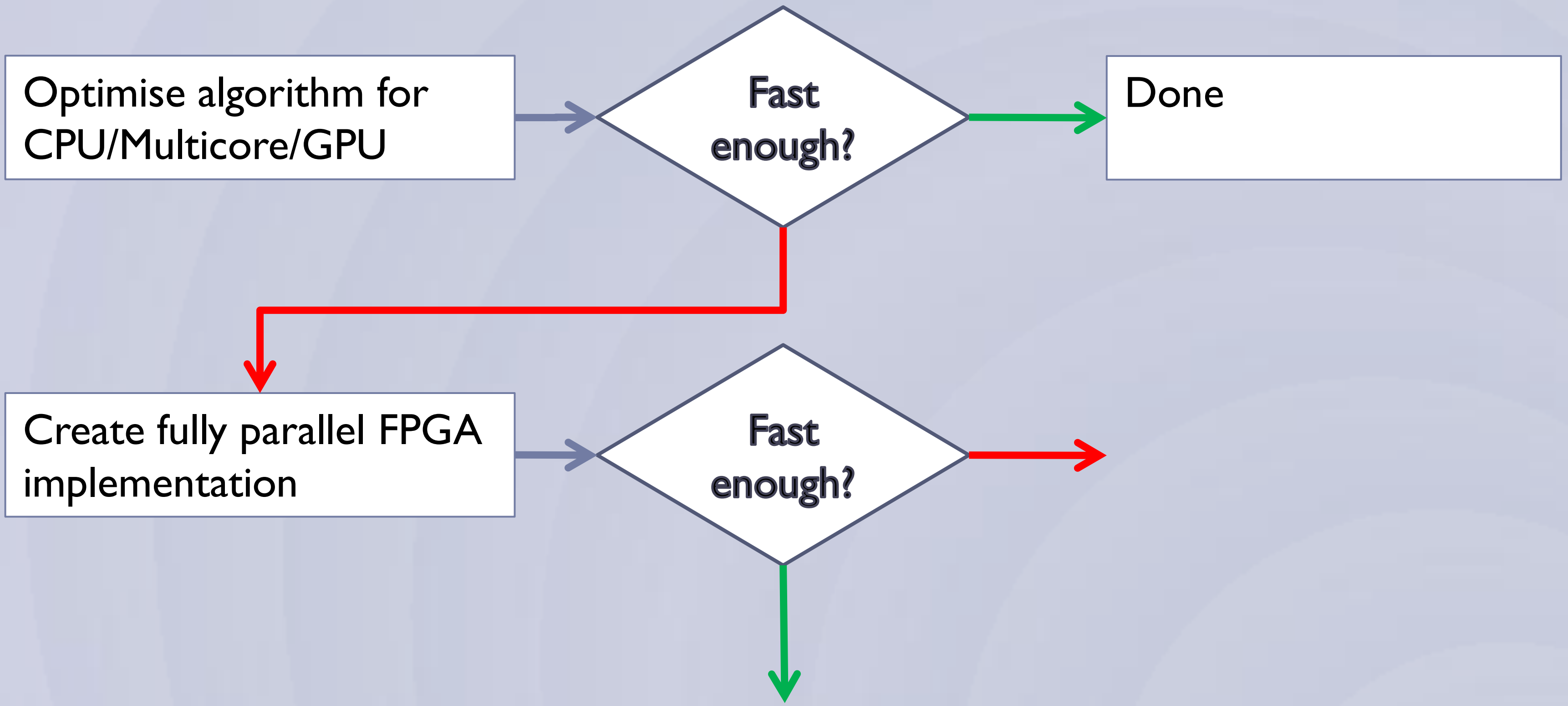

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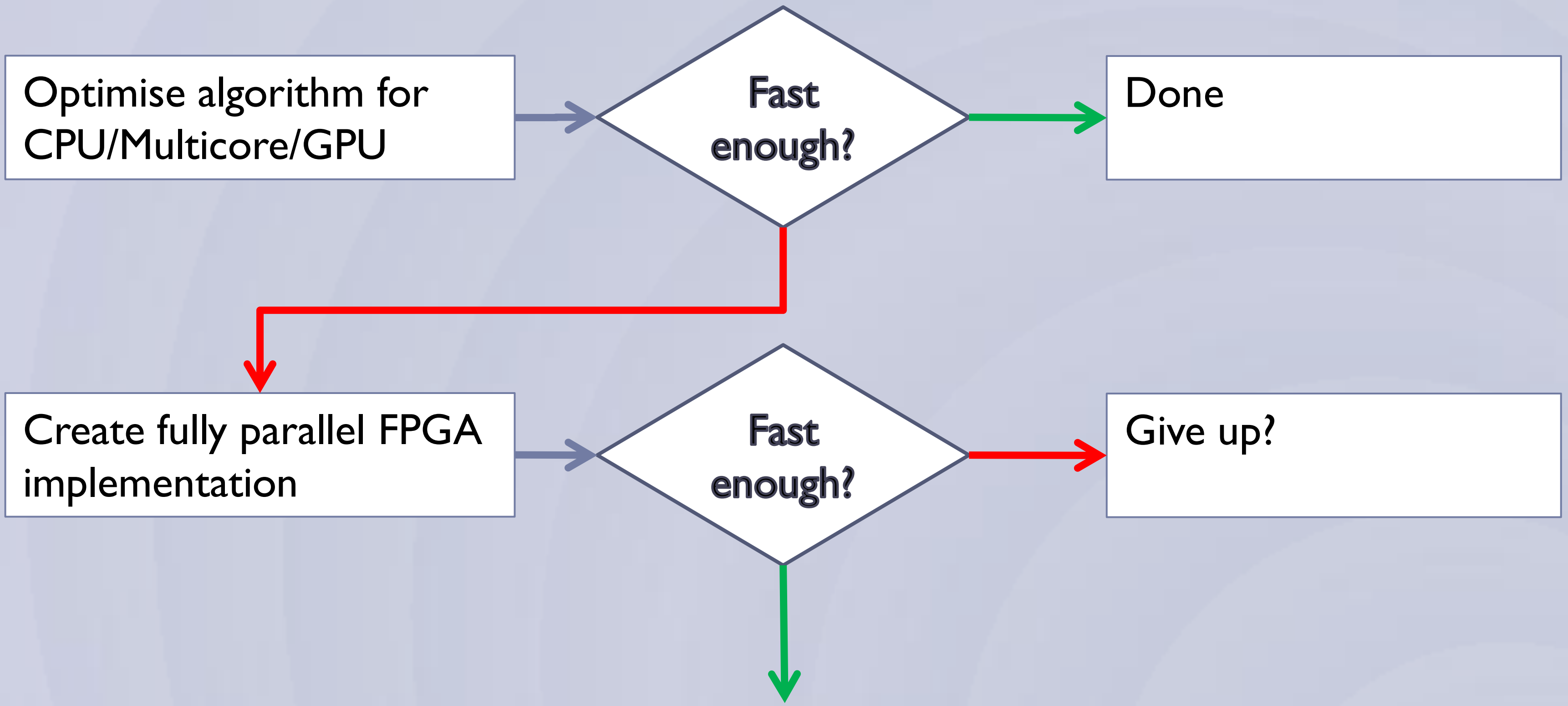

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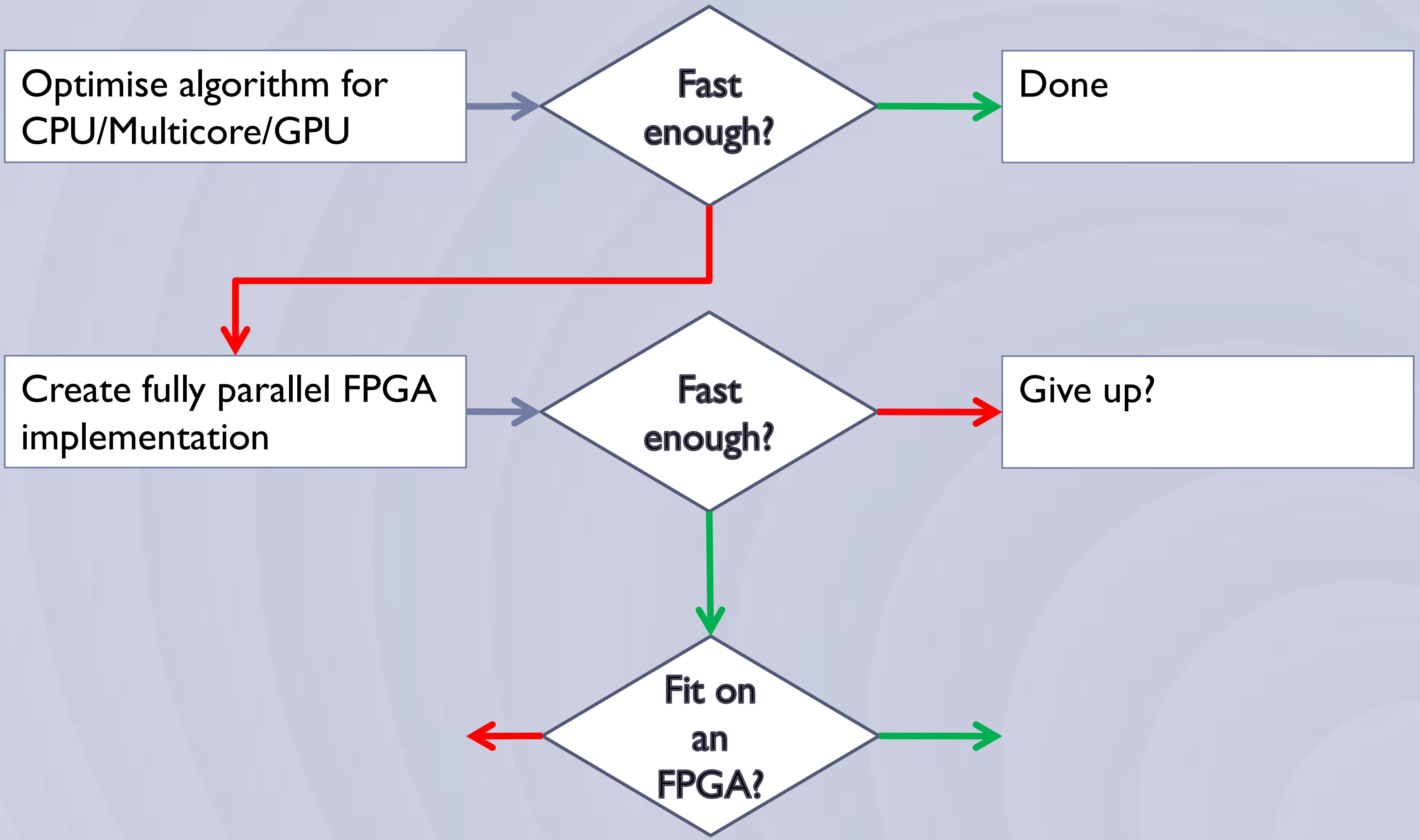

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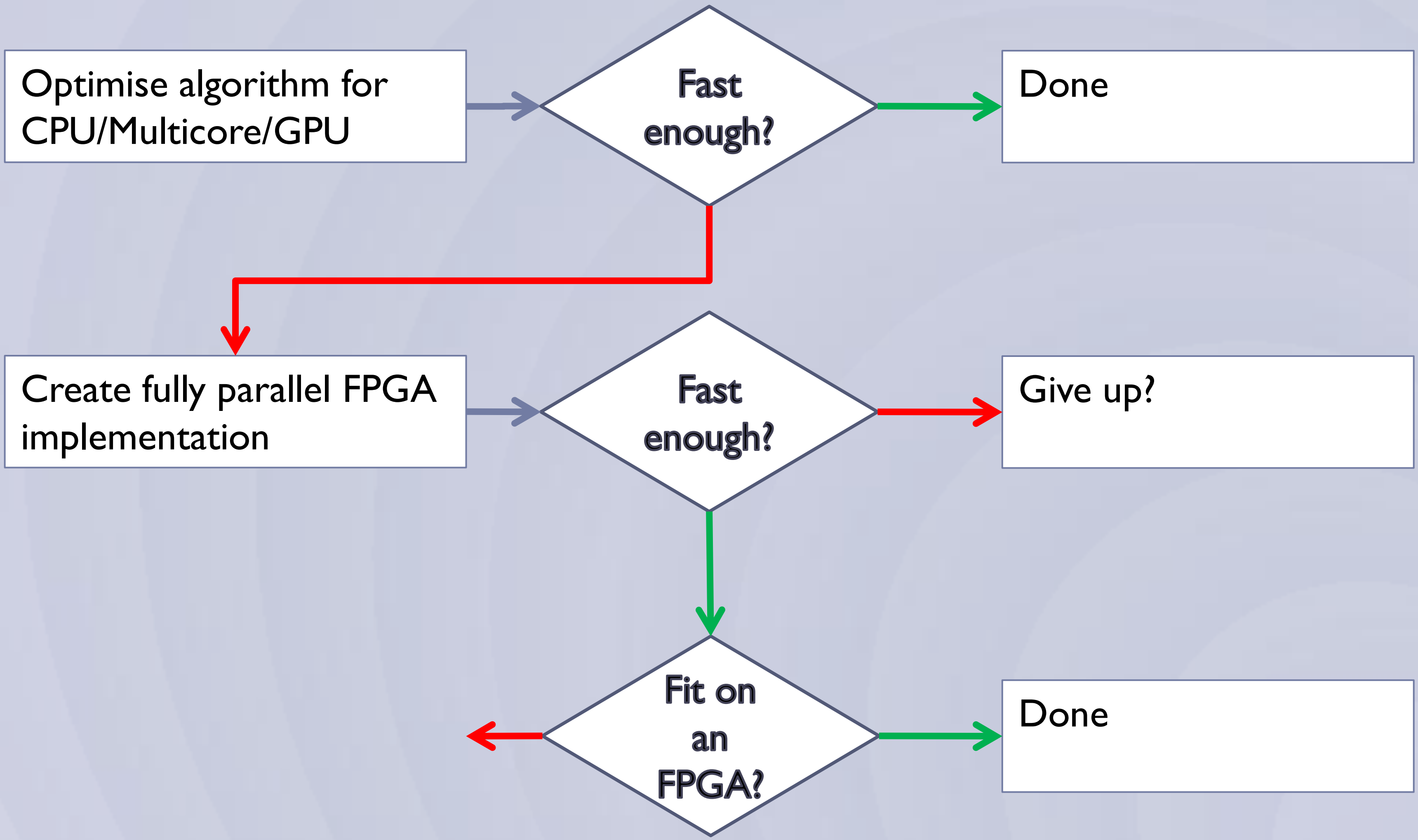

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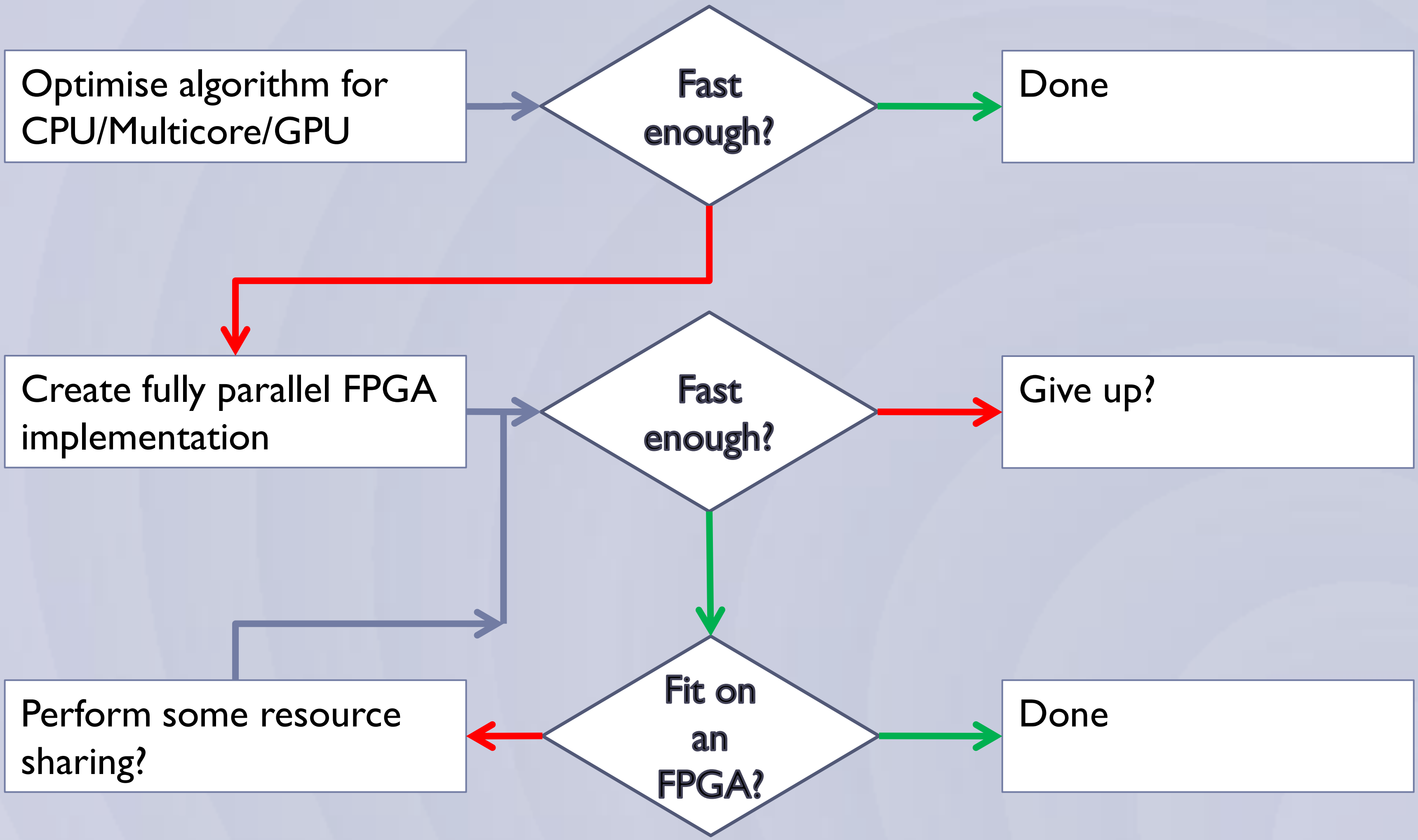

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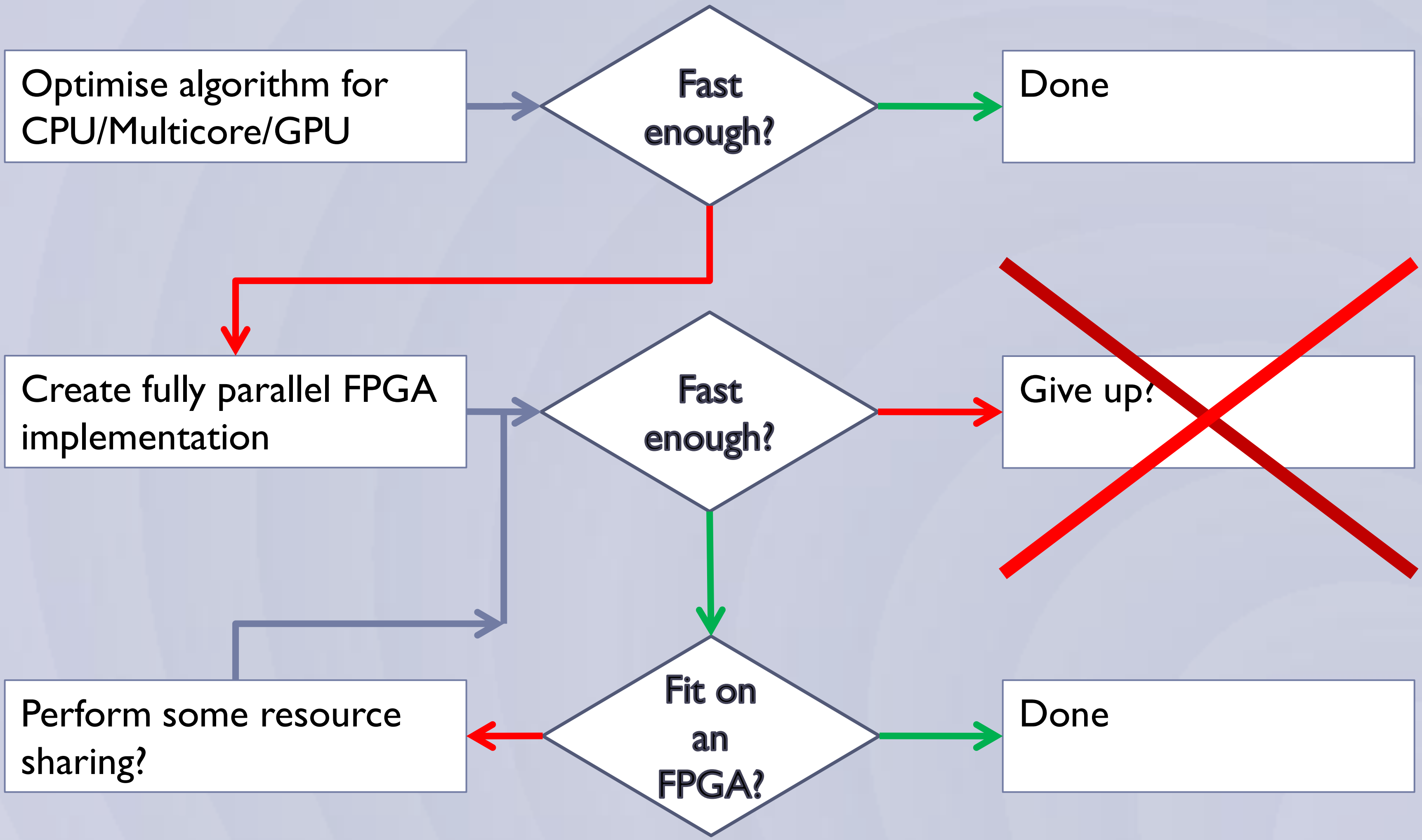

Word-length optimisation can be the game changer…

- Performance gain by moving from IEEE 754 double precision to single precision:
  - 2x for a CPU
  - 2-9x for a GPU
- FPGAs have much greater flexibility
  - Can implement any custom precision
    - Large performance trade-offs
    - Many factors affected
      - Silicon area
      - Clock speed
      - Latency
      - Memory use
      - Data transfer overhead

So why don’t we perform word-length optimisation?

- Reducing word-length can cause errors
  - Overflow error
  - Accumulation of individual round-off errors

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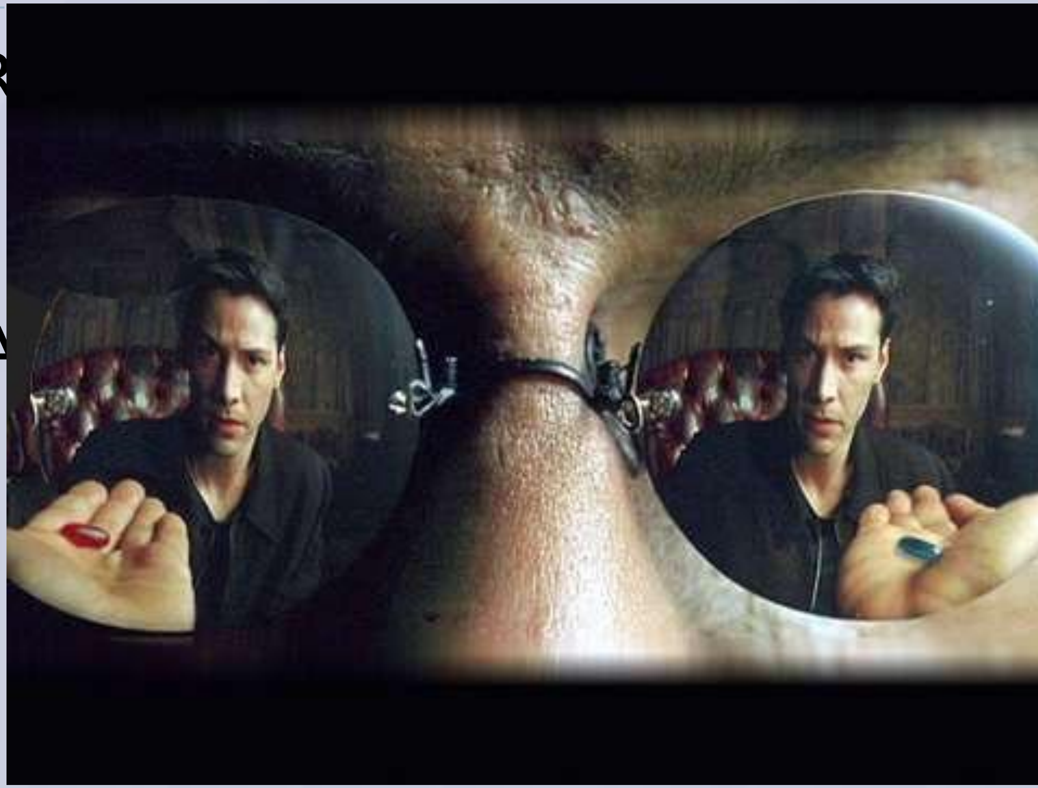
- Reducing word-length can cause errors
  - Overflow error
  - Accumulation of individual round-off errors
- Allows 'fair' comparison versus software
  - Lazy (& incorrect?) comparison
    - $(a+b)+c \neq a+(b+c)$

So why don't we perform word-length optimisation?

- R … s
- A … re



Do you want to care about numerical issues?

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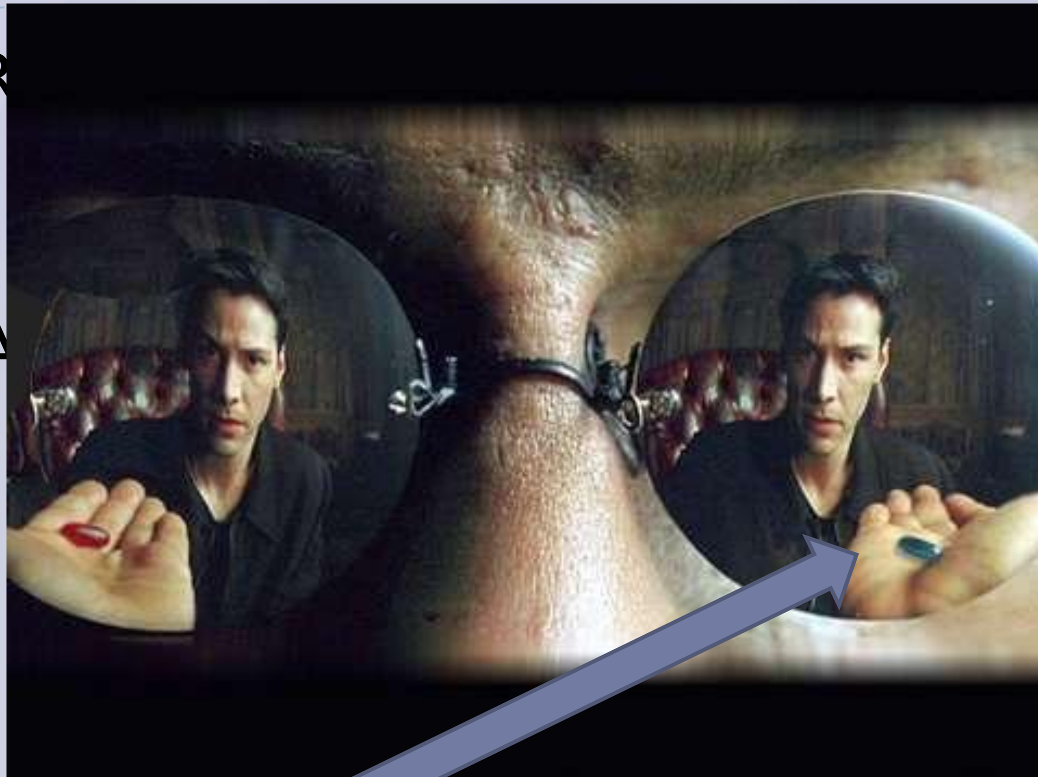

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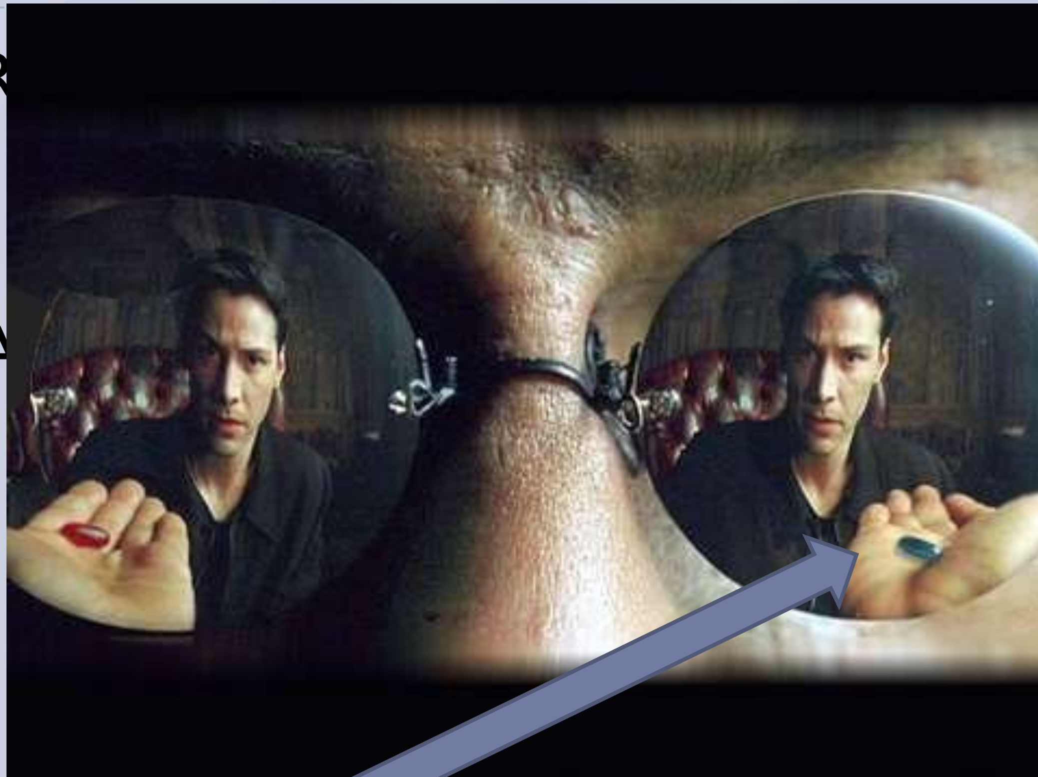


Do you want to care about numerical issues?

Report greater speed up factors by using IEEE single precision.

Ideal word-length optimisation

Algorithm being accelerated:

- Code

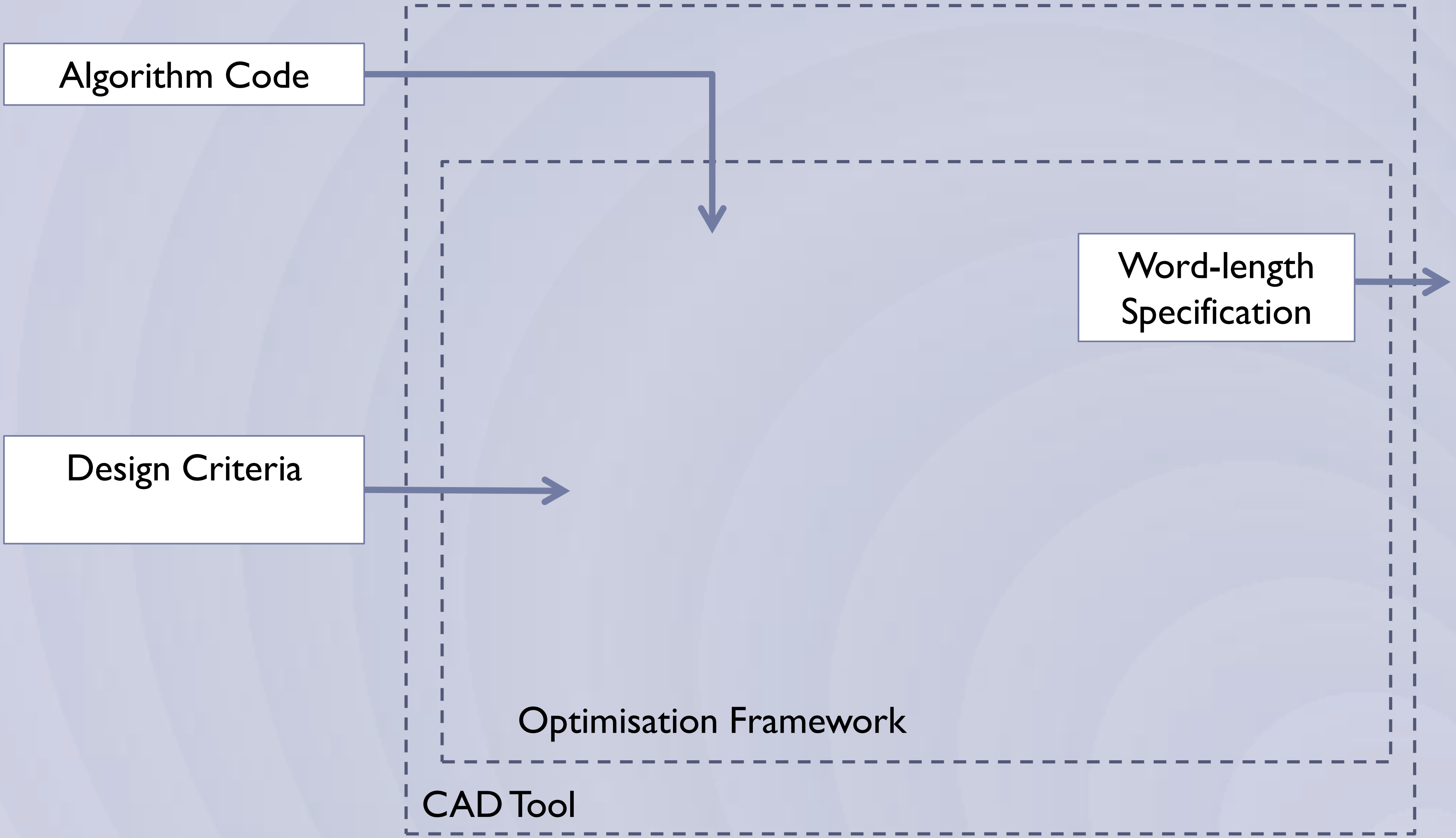
Design Criteria:

- Bound on Error
- Bound on Range

CAD Tool

Minimum word-length specification for every arithmetic operator in the hardware accelerator **guaranteed** to meet design criteria

What are the main parts of this tool?



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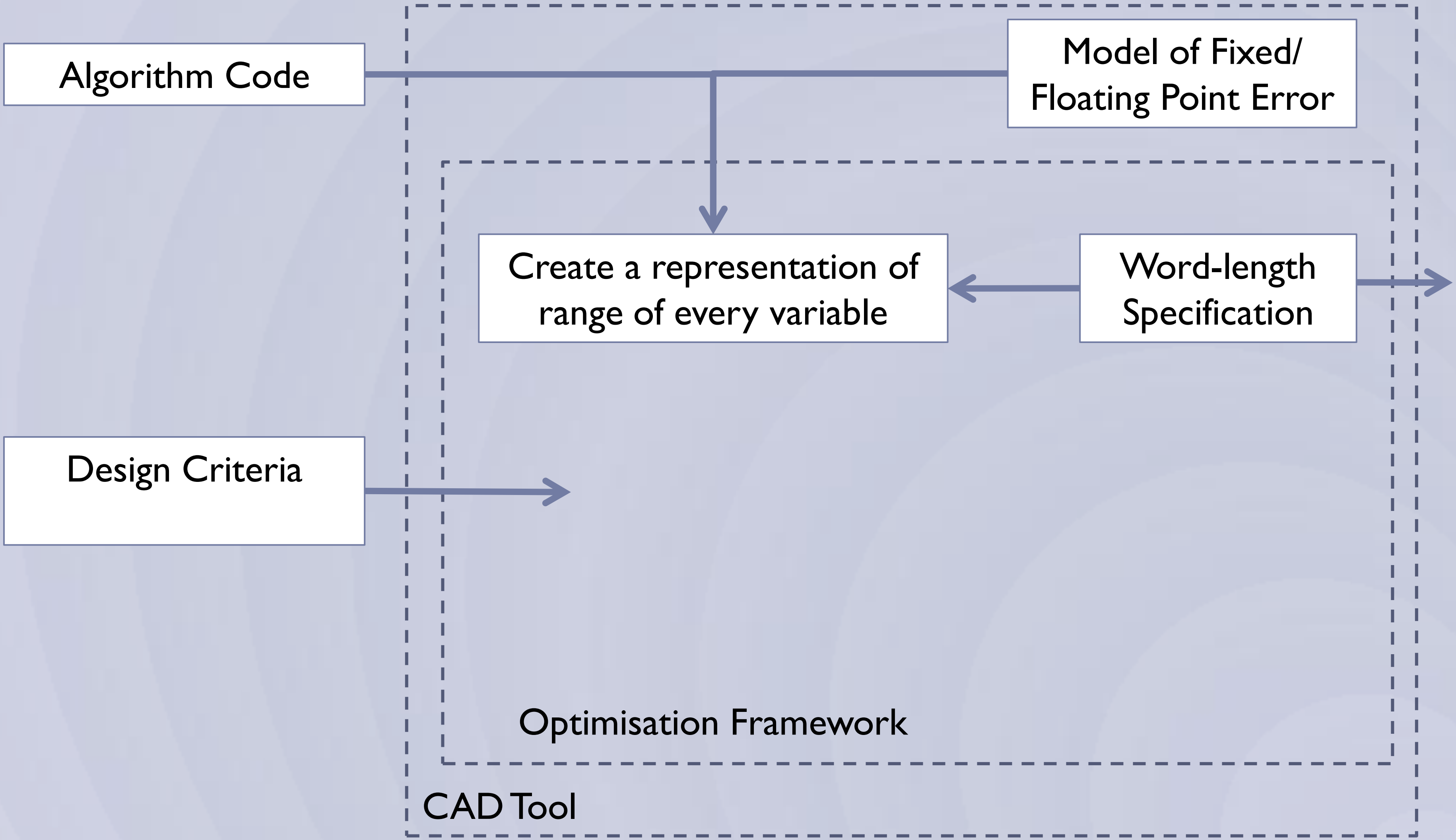

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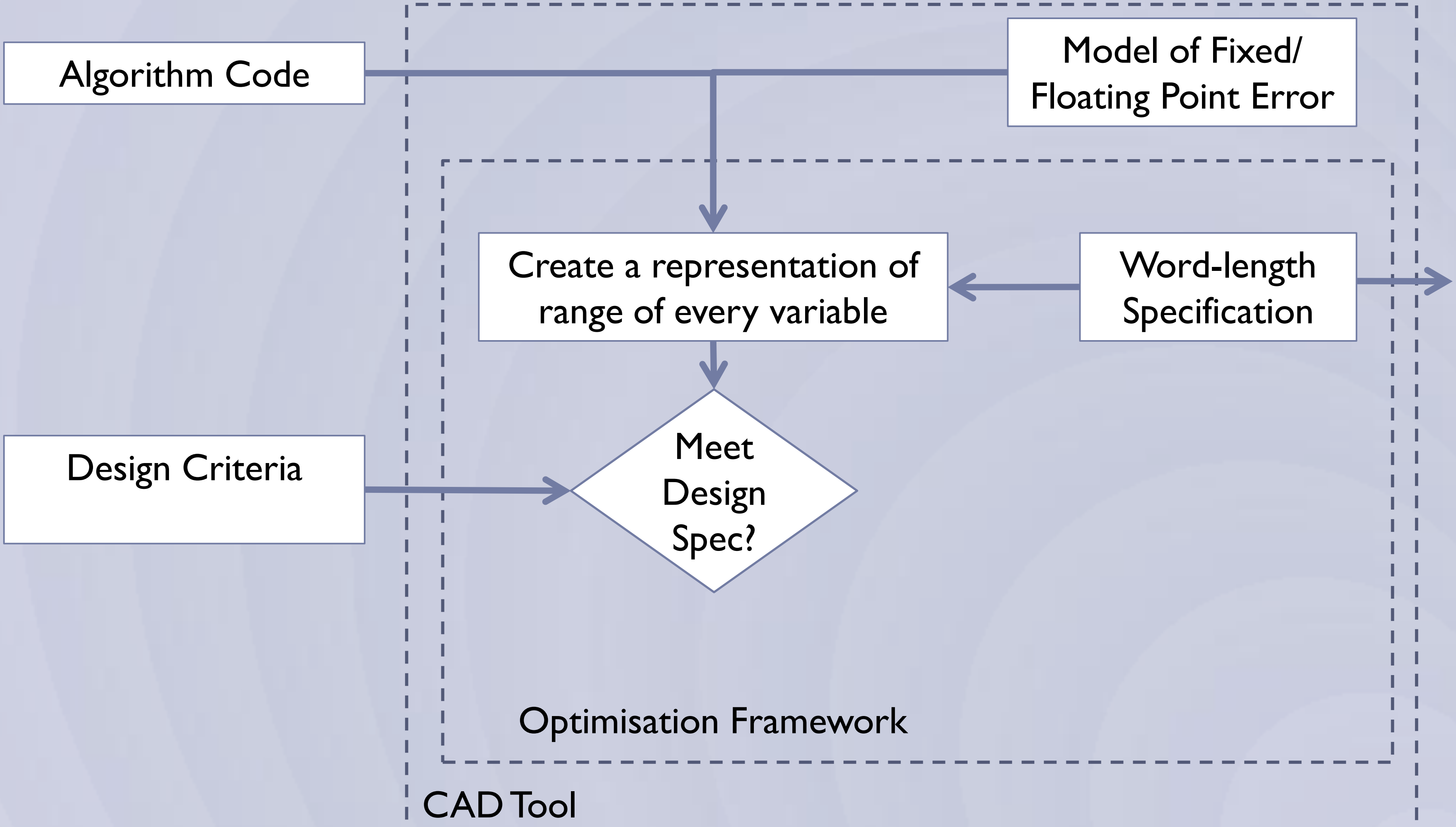

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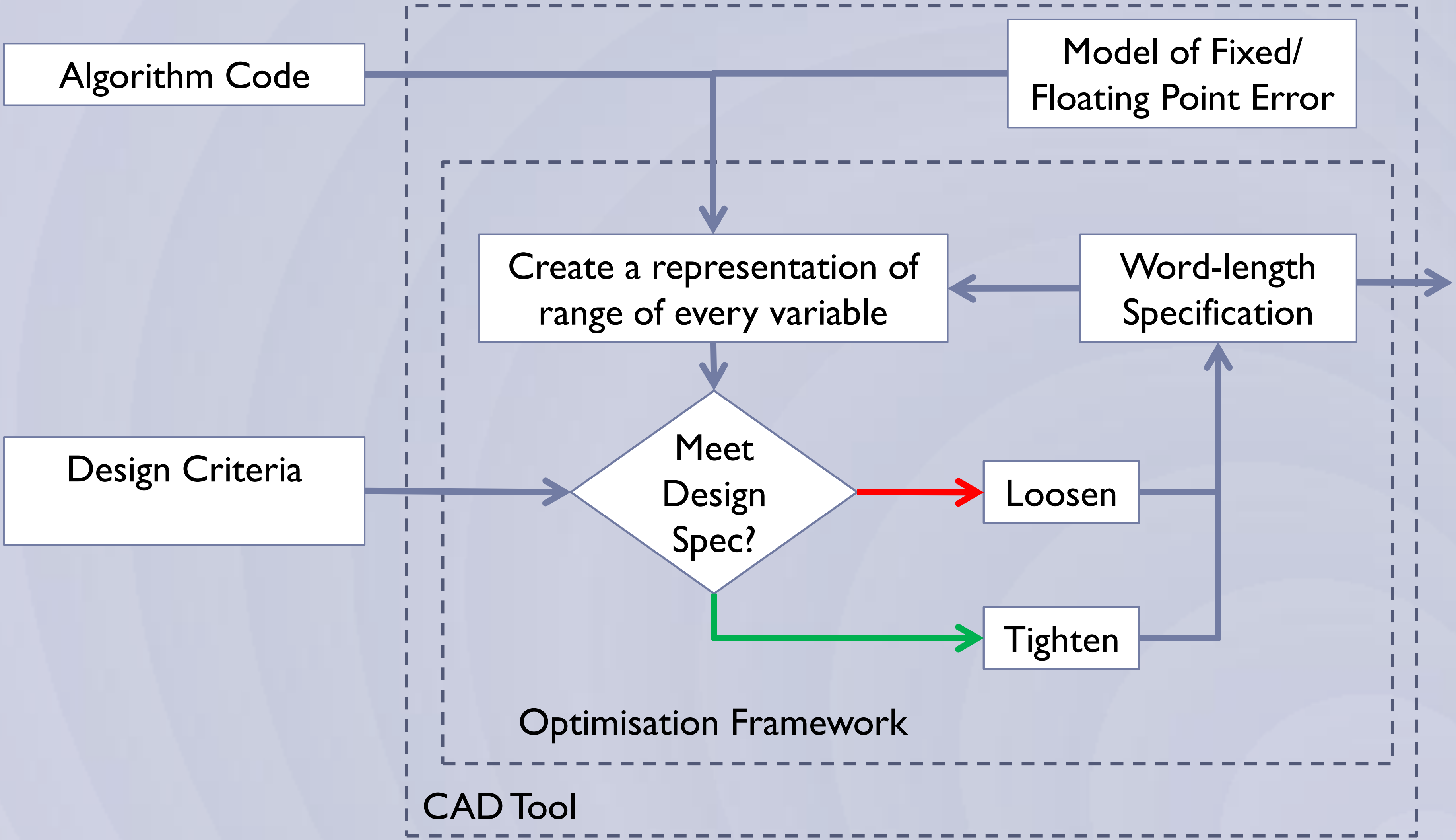

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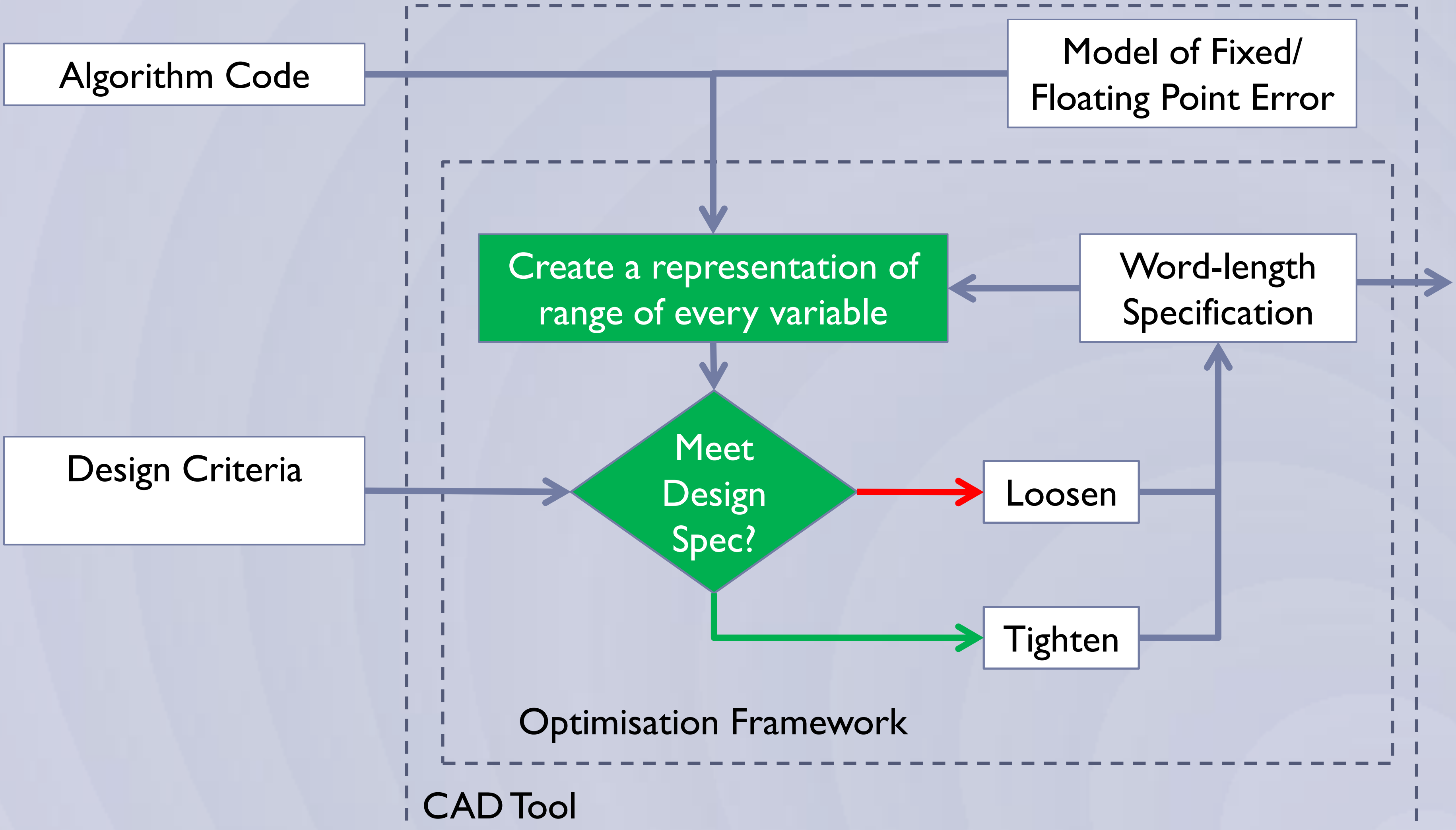
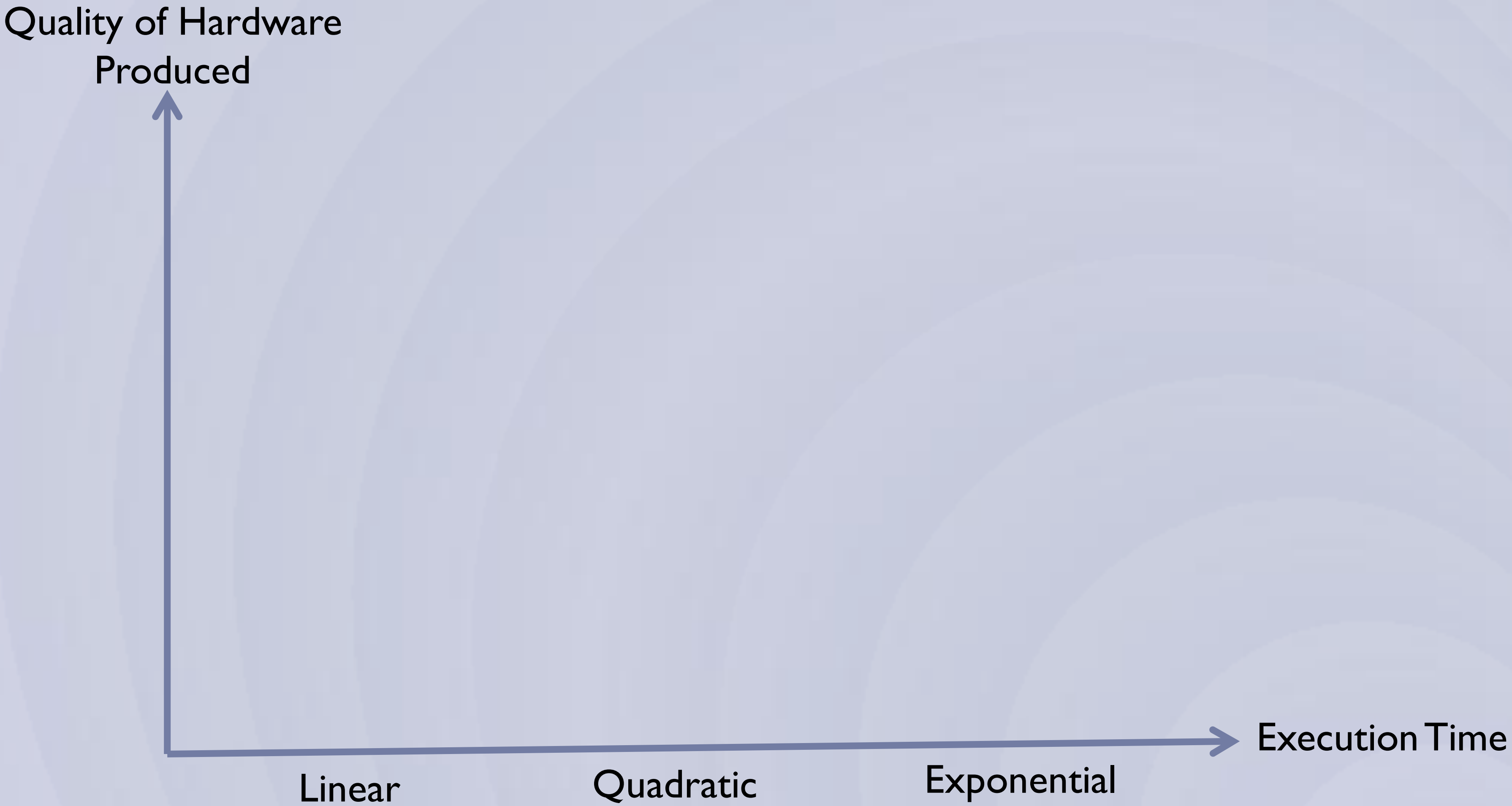


The battlefield



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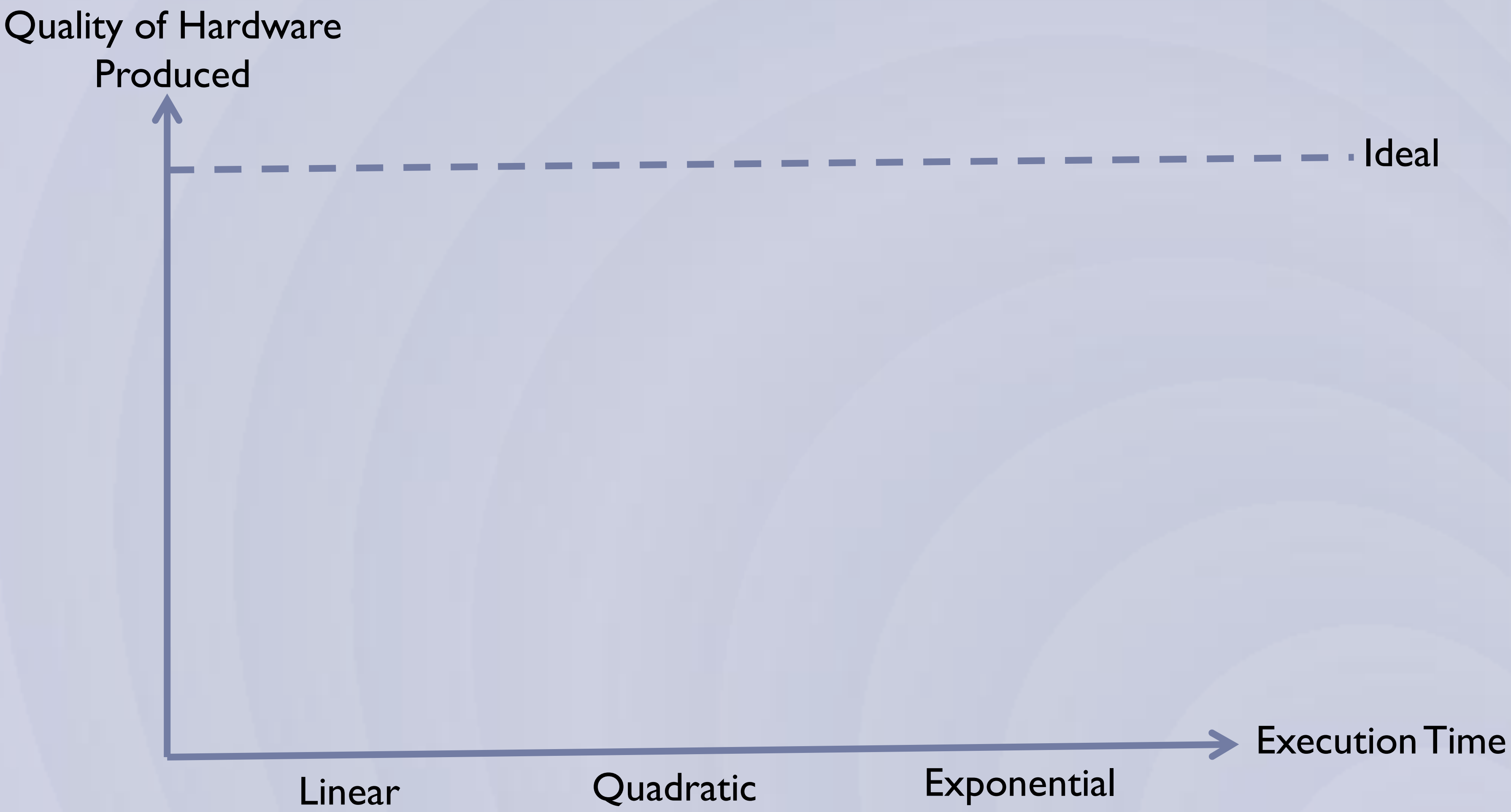

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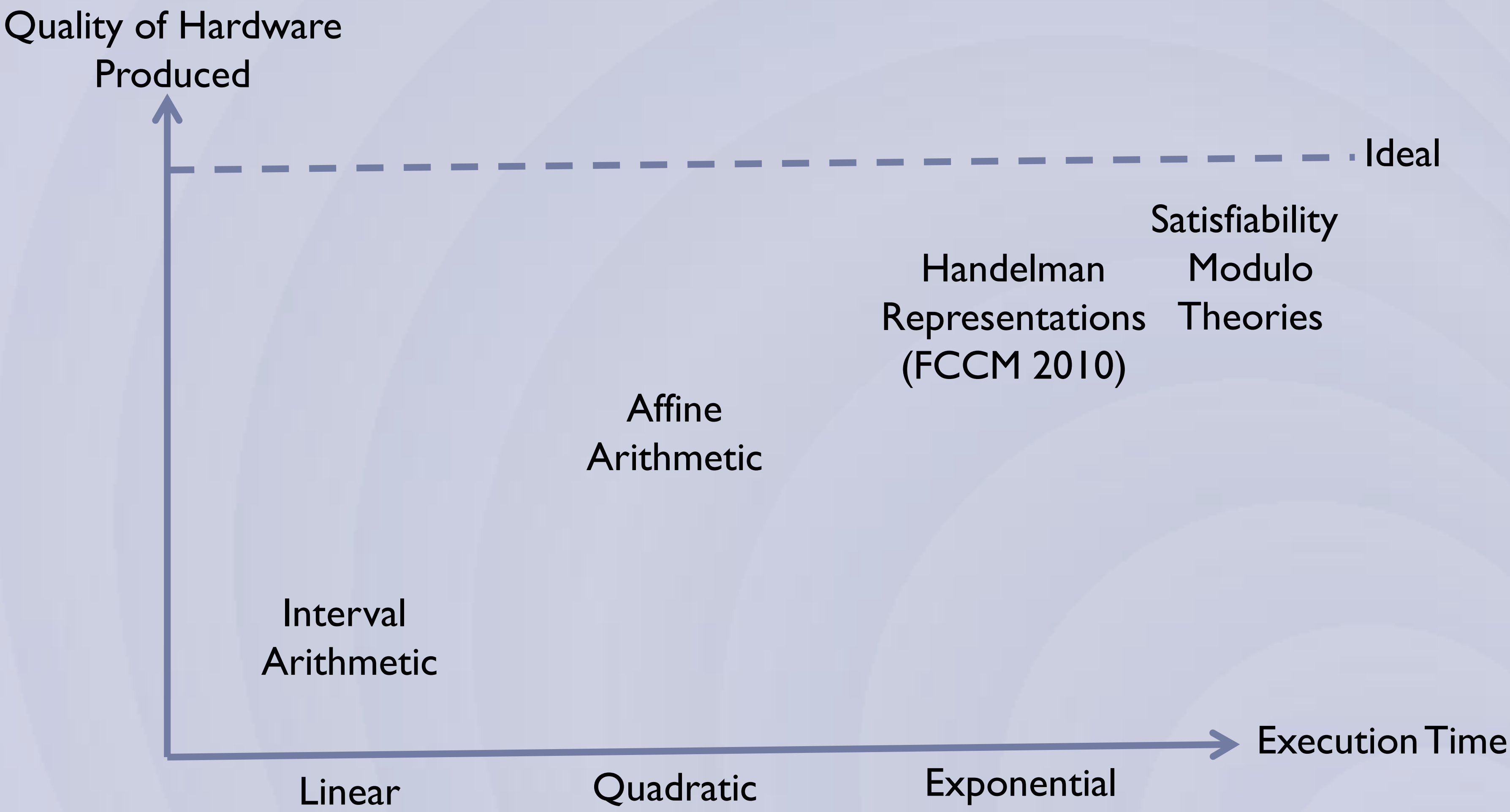

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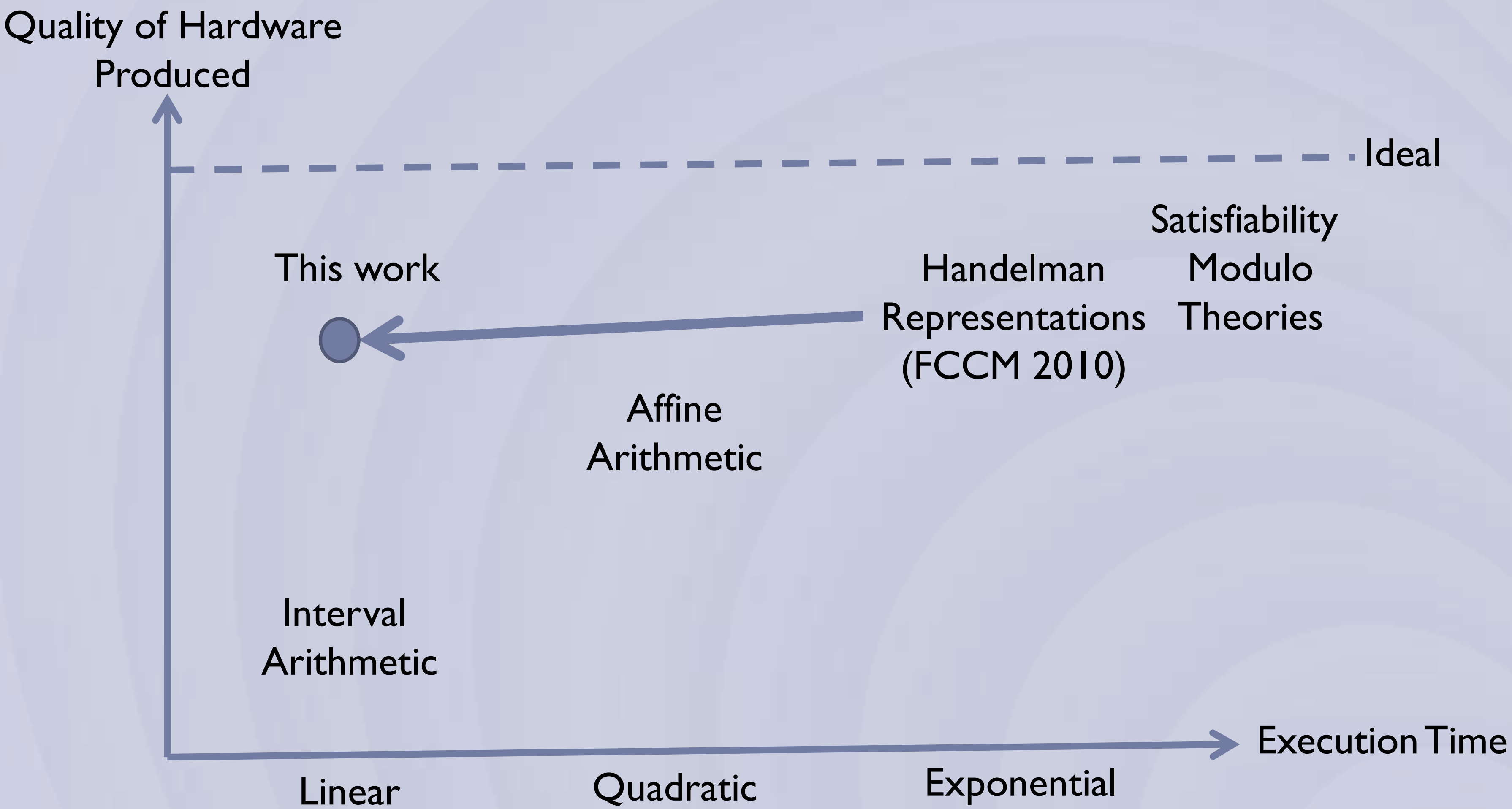
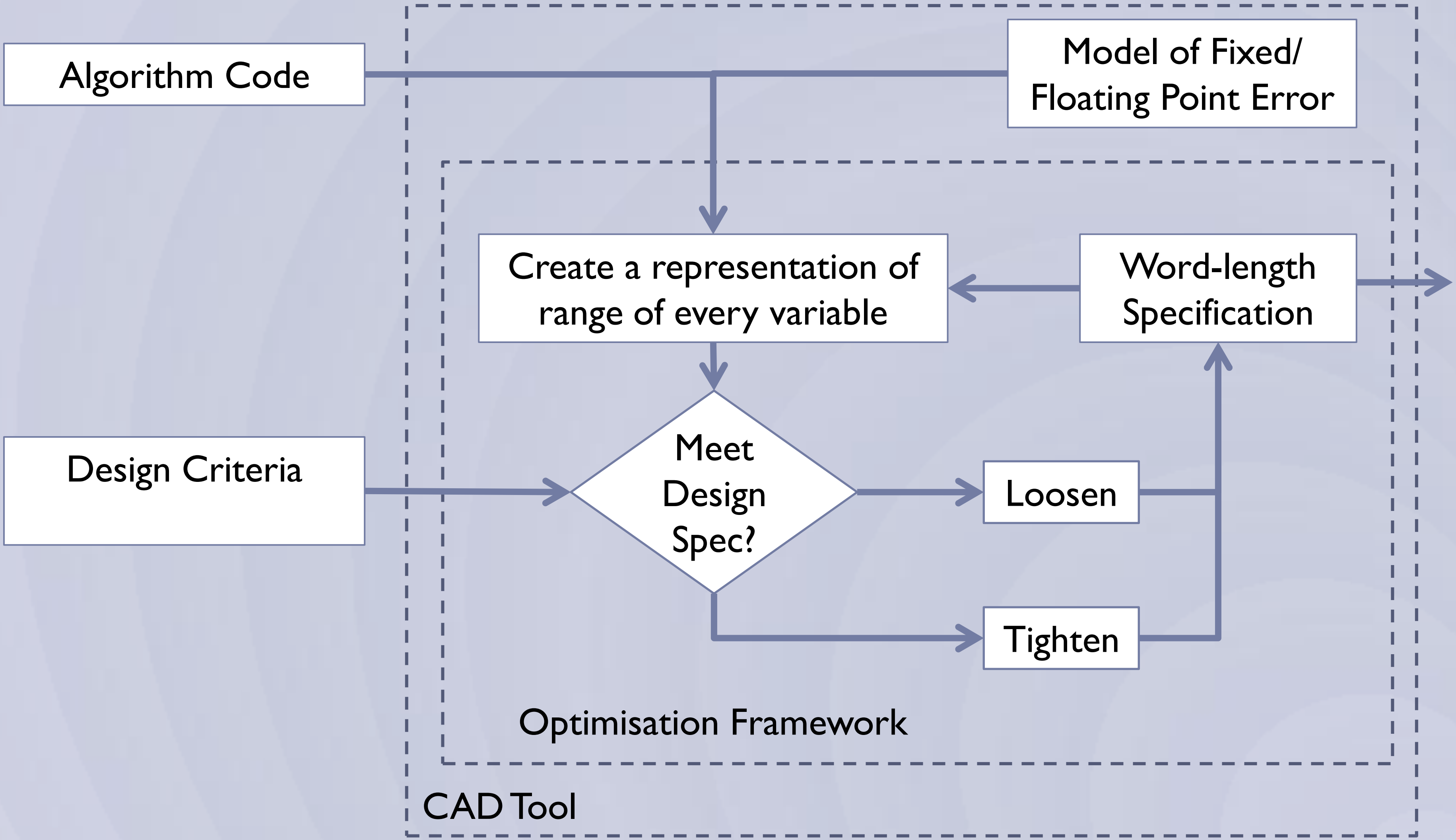


Why is scalability an issue?



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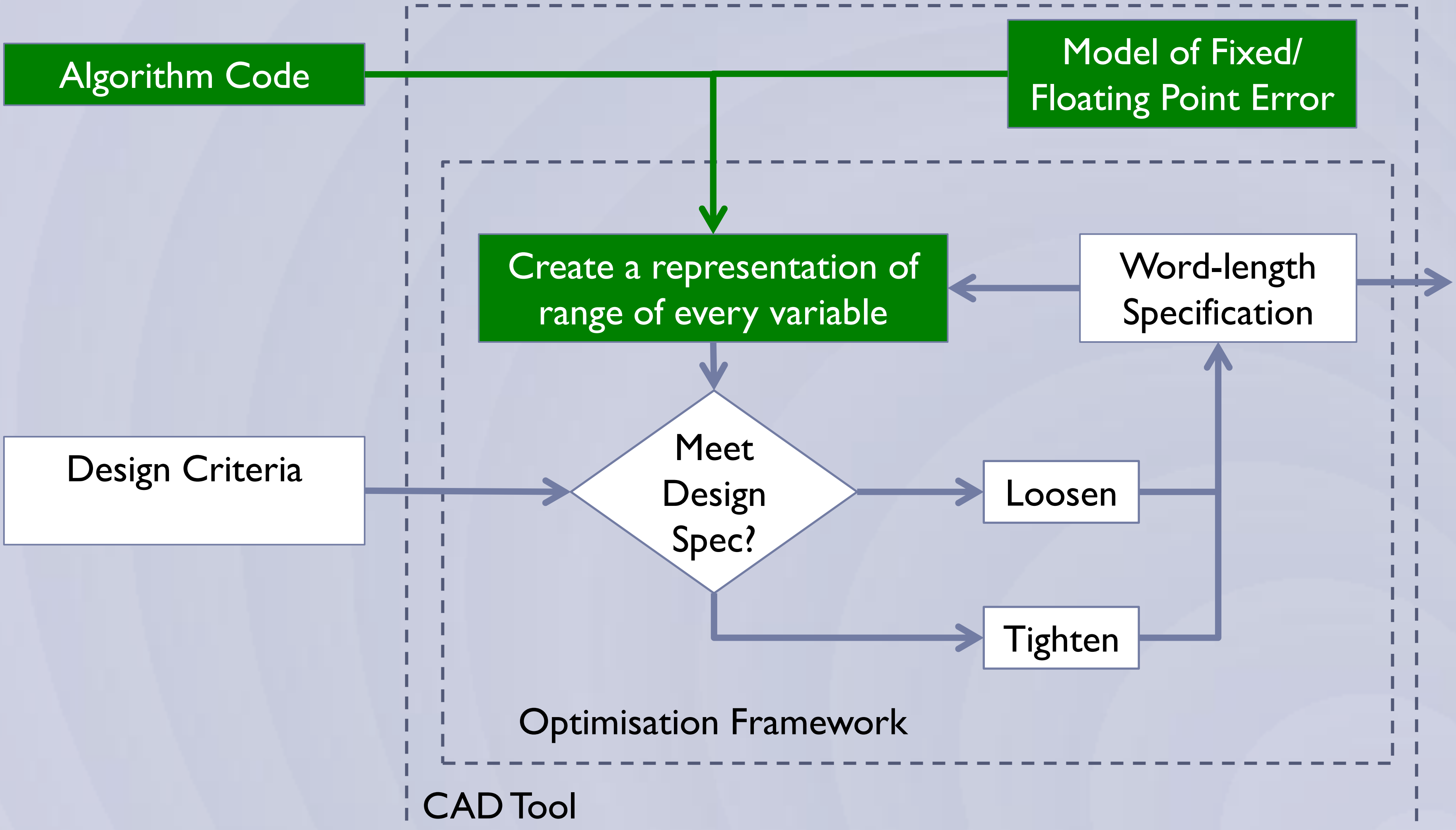

Why is scalability an issue?

- Modelling Floating Point Error
  - The closest floating point approximation $\hat{x}$ of x can be expressed as:

$$\hat{x} = x(1 + \delta_1) \qquad |\delta_1| \leq 2^{-m}$$

(m = # of mantissa bits)

  - The floating point result of any scalar operation $\odot$, where $\odot \in \{+, -, \times, \div\}$ can be bounded as:

$$\widehat{x \odot y} = (x \odot y)\,(1 + \delta_1)$$

Why is scalability an issue?

- Simple example:
  - Code: $a = x * y; b = a * z;$
  - Where: $x \in [0.8, 1.2], y \in [0.9, 1.1], z \in [9.9, 10.1]$

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  - If we denote: $|x_1| \leq 0.2, |y_1| \leq 0.1, |z_1| \leq 0.1 \; |\delta_i| \leq 2^{-12}$
  - Then: $x = (1 + x_1), y = (1 + y_1), z = (10 + z_1)$

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  - Then: $x = (1 + x_1), y = (1 + y_1), z = (10 + z_1)$
  - Create polynomials:

$$a = (1 + x_1)(1 + y_1)(1 + \delta_1)$$

$$a = 1 + x_1 + y_1 + x_1 y_1 + \delta_1 + x_1\delta_1 + y_1\delta_1 + x_1 y_1 \delta_1$$

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$$a = 1 + x_1 + y_1 + x_1 y_1 + \delta_1 + x_1\delta_1 + y_1\delta_1 \\ + x_1 y_1 \delta_1$$

$$b = (1 + x_1 + y_1 + x_1 y_1 + \delta_1 + x_1\delta_1 + y_1\delta_1 \\ + x_1 y_1 \delta_1)(10 + z_1)(1 + \delta_2)$$

Why is scalability an issue?

$$
\begin{aligned}
b &= 10 + 10x_1 + 10y_1 + 10x_1y_1 \\
&\quad +10\delta_1 + 10x_1\delta_1 + 10y_1\delta_1 + 10x_1y_1\delta_1 \\
&\quad + z_1 + x_1z_1 + y_1z_1 + x_1y_1z_1 \\
&\quad +\delta_1z_1 + x_1\delta_1z_1 + y_1\delta_1z_1 + x_1y_1\delta_1z_1 \\
&\quad + 10\delta_2 + 10x_1\delta_2 + 10y_1\delta_2 + 10x_1y_1\delta_2 \\
&\quad +10\delta_1\delta_2 + 10x_1\delta_1\delta_2 + 10y_1\delta_1\delta_2 + 10x_1y_1\delta_1\delta_2 \\
&\quad + z_1\delta_2 + x_1z_1\delta_2 + y_1z_1\delta_2 + x_1y_1z_1\delta_2 \\
&\quad +\delta_1z_1\delta_2 + x_1\delta_1z_1\delta_2 + y_1\delta_1z_1\delta_2 + x_1y_1\delta_1z_1\delta_2
\end{aligned}
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\end{aligned}
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What are the bounds on the range and relative error of b?

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\begin{aligned}
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= \; & 10 + 10x_1 + 10y_1 + 10x_1y_1 \\
& +10\delta_1 + 10x_1\delta_1 + 10y_1\delta_1 + 10x_1y_1\delta_1 \\
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& +10\delta_1\delta_2 + 10x_1\delta_1\delta_2 + 10y_1\delta_1\delta_2 + 10x_1y_1\delta_1\delta_2 \\
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\end{aligned}
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What are the bounds on the range and relative error of b?

This is computing x × y × z!!

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\end{aligned}
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Can contribute $\pm 1.1920929 \times 10^{-10}$ to final range of b

Why is scalability an issue?

Can contribute ±2 to final range of b

$$
\begin{aligned}
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&\quad + z_1\delta_2 + x_1z_1\delta_2 + y_1z_1\delta_2 + x_1y_1z_1\delta_2 \\
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\end{aligned}
$$

Can contribute ±1.1920929 × 10^{-10} to final range of b

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$$
\begin{aligned}
b &\\
= \; & 10 + 10x_1 + 10y_1 + 10x_1y_1 \\
& +10\delta_1 \cancel{+ 10x_1\delta_1 + 10y_1\delta_1 + 10x_1y_1\delta_1} \\
& + z_1 + x_1z_1 + y_1z_1 + x_1y_1z_1 \\
& \cancel{+ \delta_1z_1 + x_1\delta_1z_1 + y_1\delta_1z_1 + x_1y_1\delta_1z_1} \\
& + 10\delta_2 \cancel{+ 10x_1\delta_2 + 10y_1\delta_2 + 10x_1y_1\delta_2} \\
& \cancel{+ 10\delta_1\delta_2 + 10x_1\delta_1\delta_2 + 10y_1\delta_1\delta_2 + 10x_1y_1\delta_1\delta_2} \\
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& \cancel{+ \delta_1z_1\delta_2 + x_1\delta_1z_1\delta_2 + y_1\delta_1z_1\delta_2 + x_1y_1\delta_1z_1\delta_2}
\end{aligned}
$$

$$
\begin{aligned}
b = 10 + 10x_1 + 10y_1 + 10x_1y_1 + 10\,\delta_1 + z_1 \\
+ x_1z_1 + y_1z_1 + x_1y_1z_1 + 10\delta_2
\end{aligned}
$$

Why is scalability an issue?

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\begin{aligned}
b &= 10 + 10x_1 + 10y_1 + 10x_1y_1 \\
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\end{aligned}
$$

$$
\begin{aligned}
b = 10 + 10x_1 + 10y_1 + 10x_1y_1 + 10\,\delta_1 + z_1 \\
+ x_1z_1 + y_1z_1 + x_1y_1z_1 + 10\delta_2 + \zeta_1
\end{aligned}
\qquad |\zeta_1| \le 0.0015
$$

An added bonus

- Can use methods from *approximation theory* to make our technique applicable to algorithms including any elementary functions (e.g. Sine/Cosine/Sqrt)
  - These methods approximate an elementary function using a polynomial and an extra term bounding the error of the approximation

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&\quad \cancel{+ z_1\delta_2 + x_1z_1\delta_2 + y_1z_1\delta_2 + x_1y_1z_1\delta_2} \\
&\quad \cancel{+ \delta_1z_1\delta_2 + x_1\delta_1z_1\delta_2 + y_1\delta_1z_1\delta_2 + x_1y_1\delta_1z_1\delta_2}
\end{aligned}
$$

$$
\begin{aligned}
b = 10 + 10x_1 + 10y_1 + 10x_1y_1 + 10\,\delta_1 + z_1 \\
+ x_1z_1 + y_1z_1 + x_1y_1z_1 + 10\delta_2 + \zeta_1 \qquad |\zeta_1| \le 0.0015
\end{aligned}
$$

An added bonus

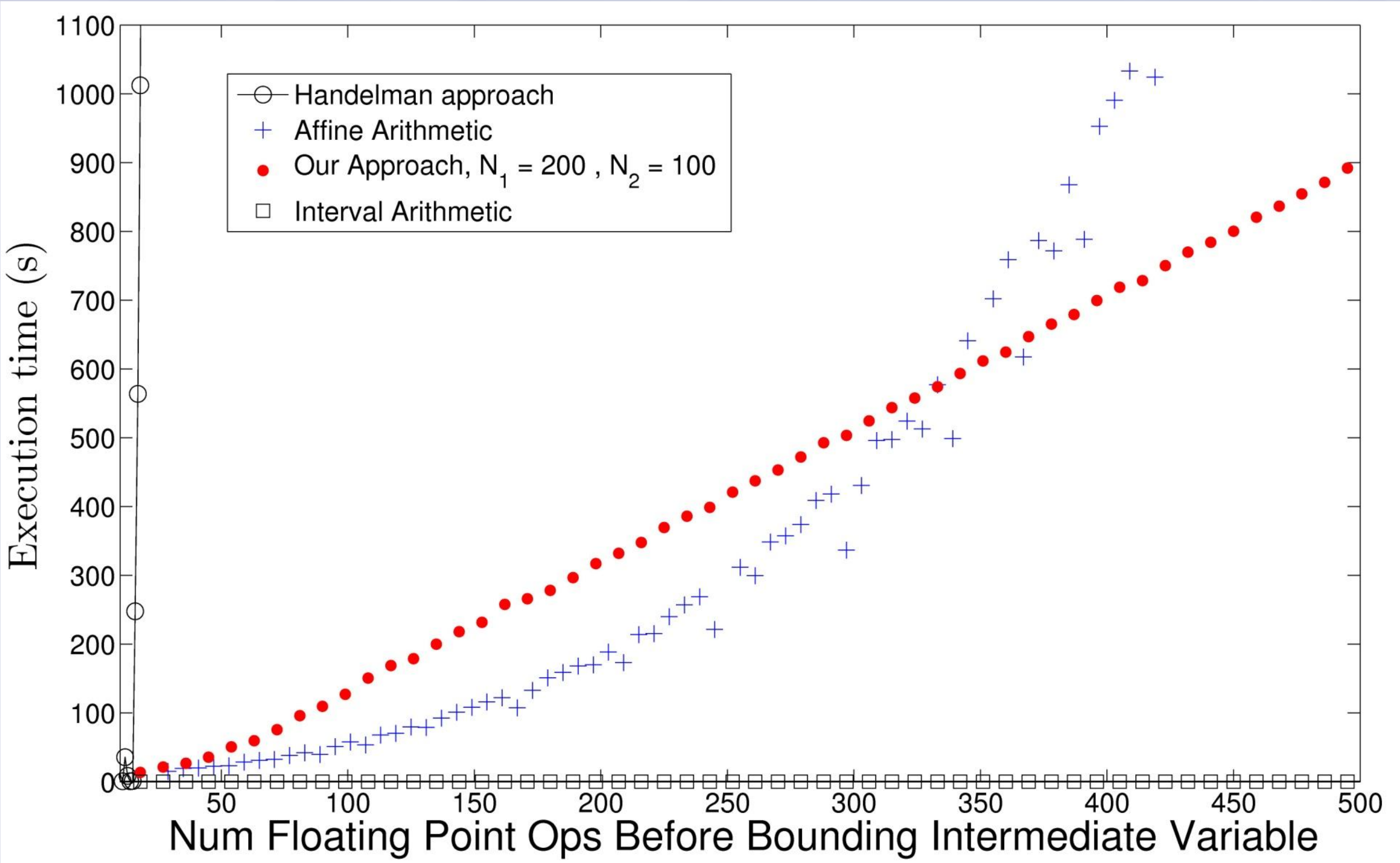
- Can use methods from *approximation theory* to make our technique applicable to algorithms including any elementary functions (e.g. Sine/Cosine/Sqrt)
  - Methods from approximation theory approximate an elementary function using a polynomial and an extra term bounding the error of the approximation
    - We get this for free!!

Tests

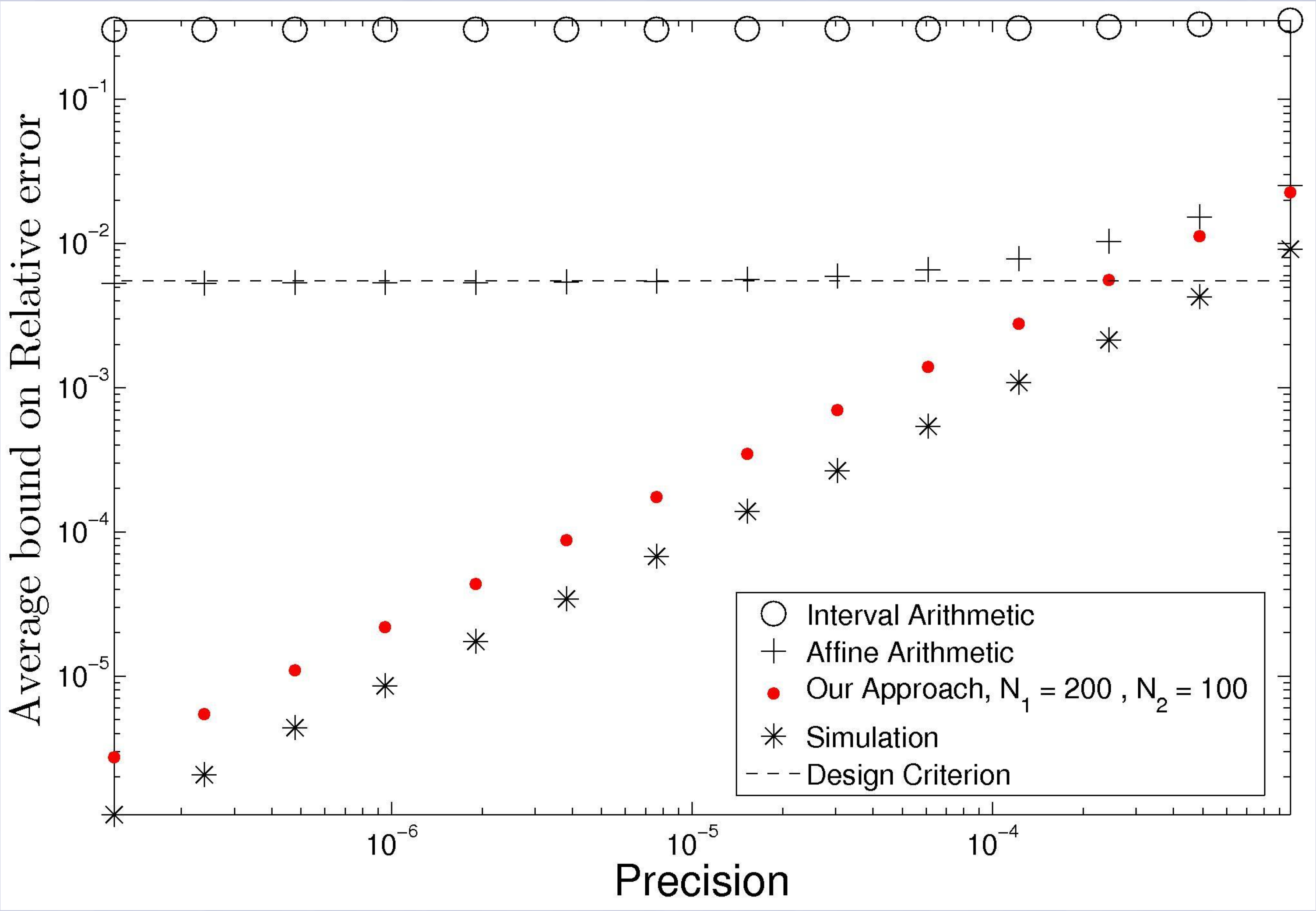
- 5x5 Successive over relaxation
  - Real algorithm to find the solution to a system of linear equations of form $Ax = b$

$$
\begin{array}{l}
\textbf{for } k = 1; k \leq 8; k++ \textbf{ do} \\
\quad \textbf{for } j = 1; j \leq 5; j++ \textbf{ do} \\
\quad\quad x^j = (1 - w)x^j + \frac{w}{A(j)^j}\left(b_j - \sum_{i=1, i \neq j}^{5} A(j)^i x^i\right) \\
\quad \textbf{end for} \\
\textbf{end for}
\end{array}
$$

Scalability: Execution time vs #operations



Quality of bounds: Relative error vs precision



Hardware use

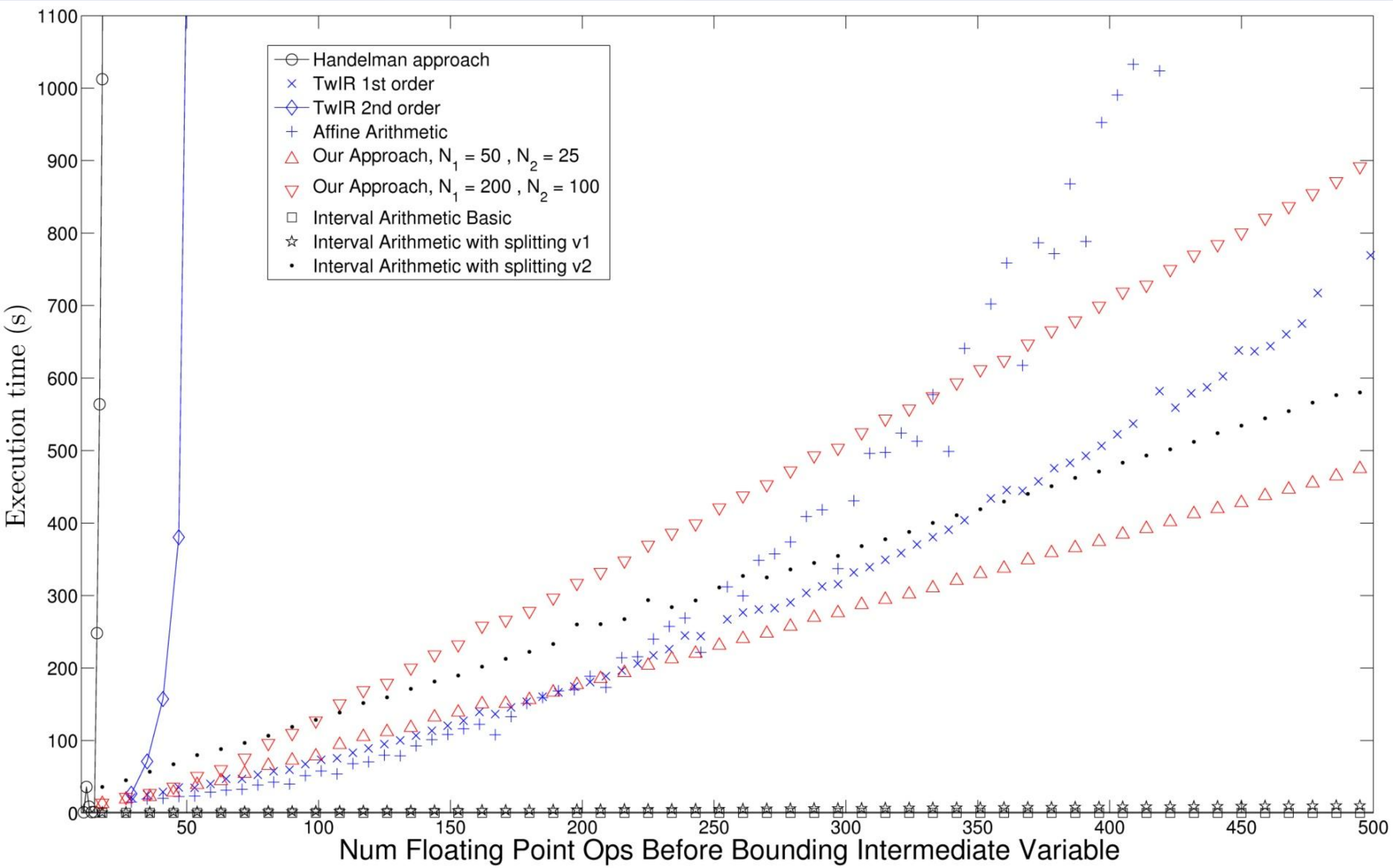
Method	Exponent (# bits)	Mantissa (# bits)	Slice Regs	Slice LUTs	Frequency (MHz)
Simulation	8	11	3562	3012	330
Our Approach	8	13	4261	3647	330
Affine Arithmetic	8	18	6606	5368	300
IA	∞	∞	∞	∞	N/A
IEEE Single Precision	8	24	8407	6815	280
IEEE Double Precision	11	53	27200	22066	251

Summary

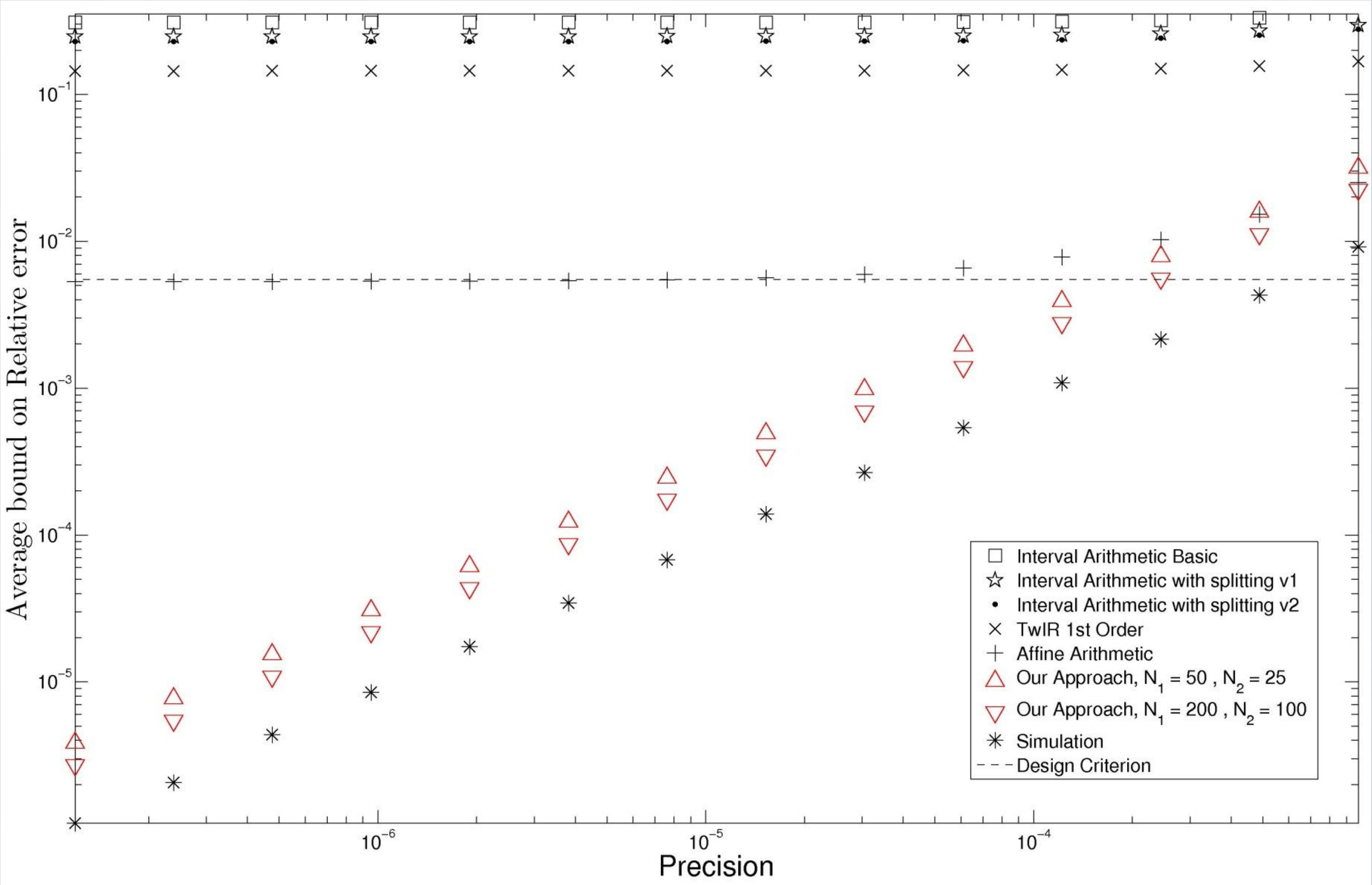
- Word-length optimisation can significantly improve hardware
- Need scalable analysis techniques to apply word-length optimisation on larger, more complex algorithms
- Our paper describes a simple set of algorithms to obtain tight bounds within a scalable execution time
  - Can use >80% fewer slice registers than IEEE double precision arithmetic
  - Can use >30% fewer slice registers than competing methods.
  - Can create hardware that is guaranteed to meet design criteria that is not possible using alternative methods

Thank you for listening

Scalability: Execution time vs #operations



Quality of bounds: Relative error vs precision



Quality of bounds vs execution time