



# An Efficient FPGA Implementation of QR Decomposition using a Novel Systolic Array Architecture based on Enhanced Vectoring CORDIC

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# Structure of Our Presentation

- Background of Our Research
- What did we do?
- Future Work

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# Background of Our Research

- MIMO-OFDM are chosen in IEEE 802.11n, LTE, WiMAX ...
- QR decomposition plays an important role in MIMO-OFDM systems

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# QR Decomposition

- Gram-Schmidt (GS) : Speed is restricted by the multiply-accumulator
- Householder Transformation (HT) : The architecture is very complicated and consumes more hardware resources
- Givens Rotation (GR) : Better accuracy and can be easily implemented in a systolic array architecture

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# Givens Rotation (GR)

- The elementary Givens matrix:

$$T_{ij} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & c & & s \\ & & & & 1 & \\ & & & & & \ddots & \\ & & & -s & & 1 & c \\ & & & & & & 1 \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{bmatrix}$$

- Assuming that

$$\begin{cases} X = [x_1, x_2, x_3, \dots, x_n]^T \\ Y = [y_1, y_2, y_3, \dots, y_n]^T = T_{ij} \cdot X \end{cases}$$



$$\begin{cases} y_i = c \cdot x_i + s \cdot x_j \\ y_j = -s \cdot x_i + c \cdot x_j \\ y_k = x_k \quad (k \neq i, j) \end{cases}$$





# Givens Rotation (GR)

- Under the condition  $x_i^2 + x_j^2 \neq 0$ , we can set

$$c = \frac{x_i}{\sqrt{x_i^2 + x_j^2}}, s = \frac{x_j}{\sqrt{x_i^2 + x_j^2}}$$

- Then

$$y_i = \sqrt{x_i^2 + x_j^2}, y_j = 0$$

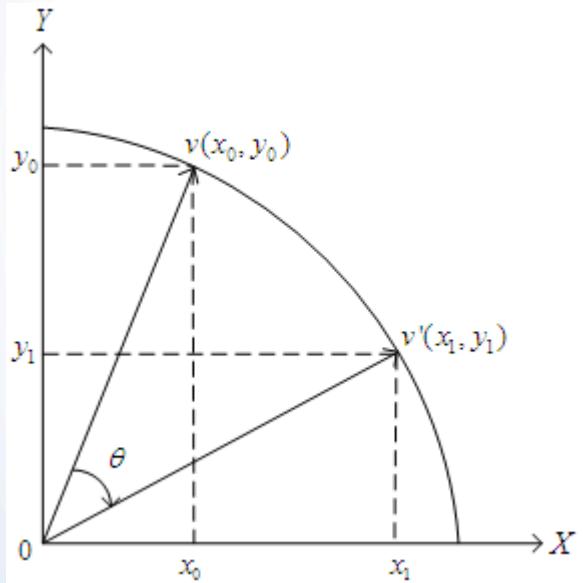
- For  $n \times n$  matrix, GR will start from zeroing the bottom elements to the upper elements of the first column, and then do the second column, and so on.





# CORDIC based GR

- In planar coordinates, the vector  $v = [x_0, y_0]^T$  is rotated by the angle  $\theta$  to get a new vector  $v' = [x_1, y_1]^T$ .



The corresponding rotation matrix is

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

As  $\cos^2 \theta + \sin^2 \theta = 1$ , then

$$c = \frac{\cos \theta}{\sqrt{\cos^2 \theta + \sin^2 \theta}}, s = \frac{\sin \theta}{\sqrt{\cos^2 \theta + \sin^2 \theta}}$$





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# What did we do?

- Enhanced Vectoring CORDIC
- A Novel Systolic Array Architecture for QR
- Validations and Experiments

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# Enhanced Vectoring CORDIC

- Scale the convergence range of CORDIC into  $[0, \pi/4]$  by reducing the i=0 iteration
- Adopt the adaptive recoding method to recode the scale compensation factor k

	$K$	$K'$
Code	16b0011011011110110	16b0100100100001010
Sign	16b000000000000000000	16b0000100100001010
Value	$K = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \frac{1}{2^{10}} + \frac{1}{2^{12}} + \frac{1}{2^{13}}$	$K' = 1 - \frac{1}{2^3} - \frac{1}{2^6} - \frac{1}{2^{11}} - \frac{1}{2^{13}}$

- Mix every two consecutive iterations into one stage to balance the pipeline



# A Novel Systolic Array Architecture for QR

- Assuming the size of the required factorization matrix  $X$  is  $n \times n$ , after multiplied by  $T_{ij}$ , we can get

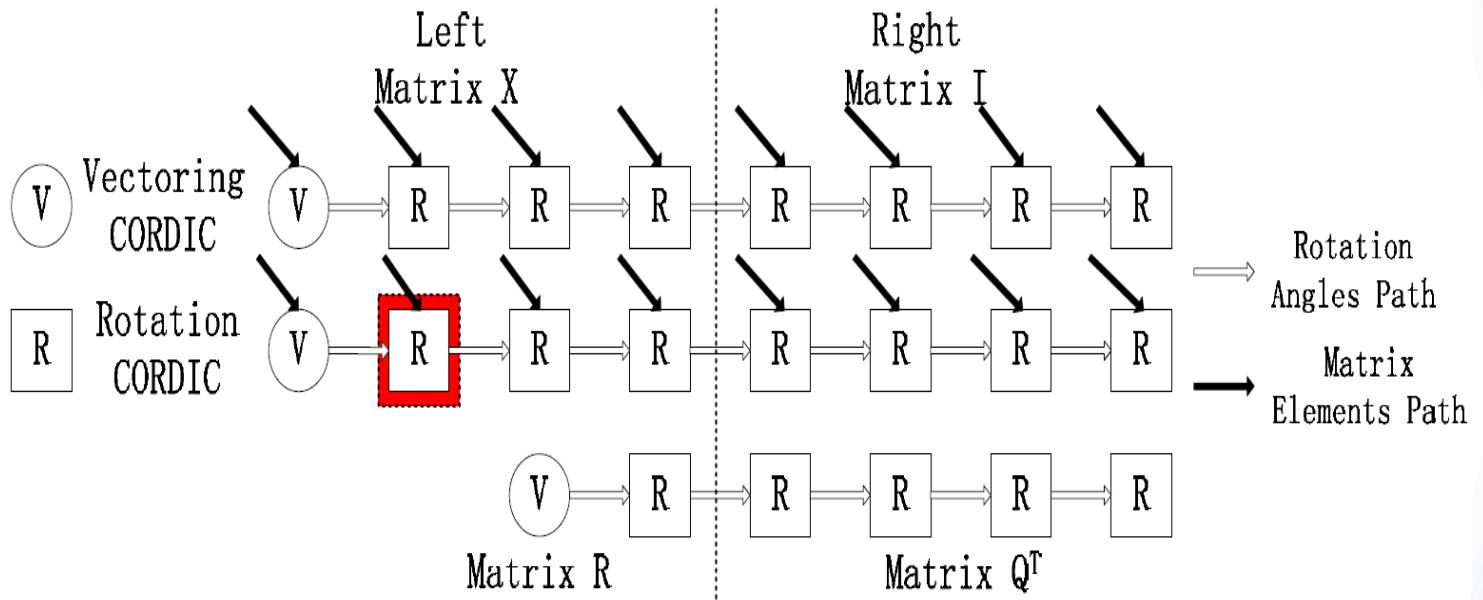
$$X' = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{i-1,1} & x_{i-1,2} & \cdots & x_{i-1,n} \\ x_{i,1} \cdot c + x_{j,1} \cdot s & x_{i,2} \cdot c + x_{j,2} \cdot s & \cdots & x_{i,n} \cdot c + x_{j,n} \cdot s \\ x_{i+1,1} & x_{i+1,2} & \cdots & x_{i+1,n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{j-1,1} & x_{j-1,2} & \cdots & x_{j-1,n} \\ -x_{i,1} \cdot s + x_{j,1} \cdot c & -x_{i,2} \cdot s + x_{j,2} \cdot c & \cdots & -x_{i,n} \cdot s + x_{j,n} \cdot c \\ x_{j+1,1} & x_{j+1,1} & \cdots & x_{j+1,1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,n} \end{bmatrix}$$





# A Novel Systolic Array Architecture for QR

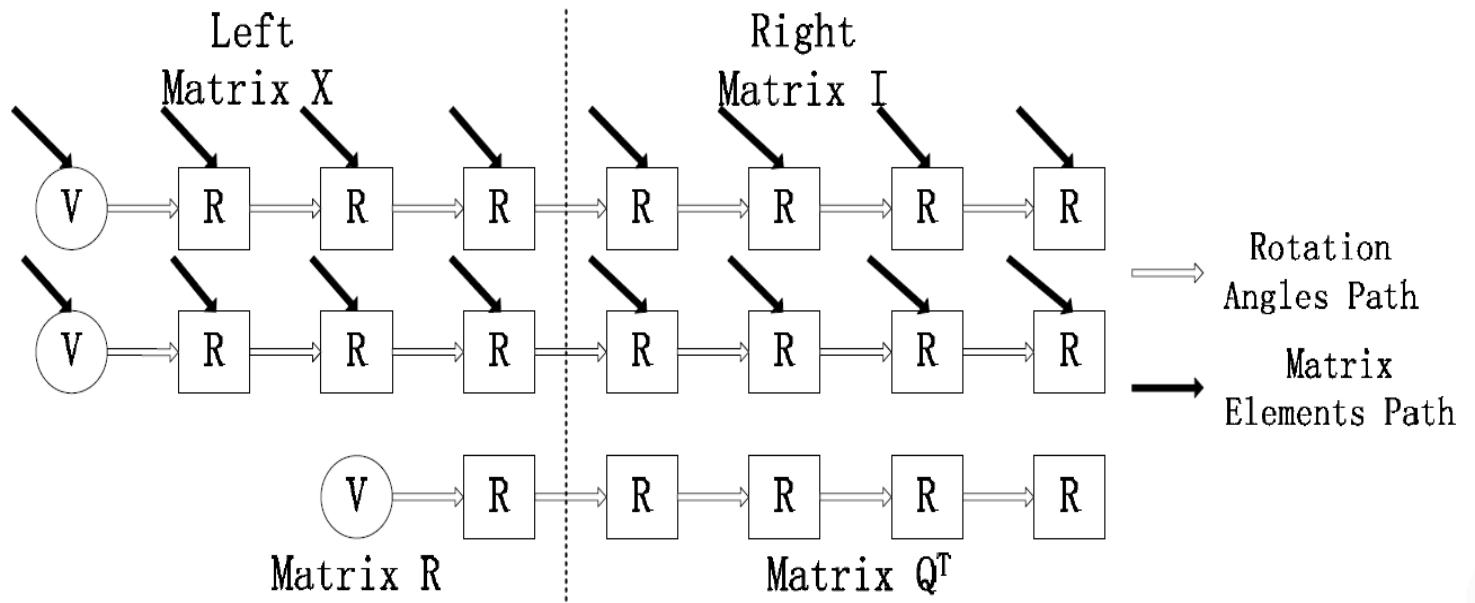
- The novel systolic array architecture for QR is





# A Novel Systolic Array Architecture for QR

- The first step:





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# QR Decomposition (4×4)

$$X = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{6} & \frac{1}{9} \\ \frac{1}{3} & \frac{3}{4} & \frac{1}{7} & \frac{1}{8} \\ \frac{2}{5} & \frac{1}{4} & \frac{1}{2} & \frac{1}{6} \\ \frac{3}{10} & \frac{1}{3} & \frac{2}{7} & \frac{1}{4} \end{bmatrix} \xrightarrow{\text{Step 1}}$$

For the first and fourth rows, zeroing  $x_{4,1}$

$$c_{4,1} = \frac{\frac{1}{4}}{\sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{3}{10}\right)^2}} \approx 0.6402 \quad s_{4,1} = \frac{\frac{3}{10}}{\sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{3}{10}\right)^2}} \approx 0.7682$$

For the second and third rows, zeroing  $x_{3,1}$

$$c_{3,1} = \frac{\frac{1}{3}}{\sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{5}\right)^2}} \approx 0.6398 \quad s_{3,1} = \frac{\frac{2}{5}}{\sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{5}\right)^2}} \approx 0.7685$$

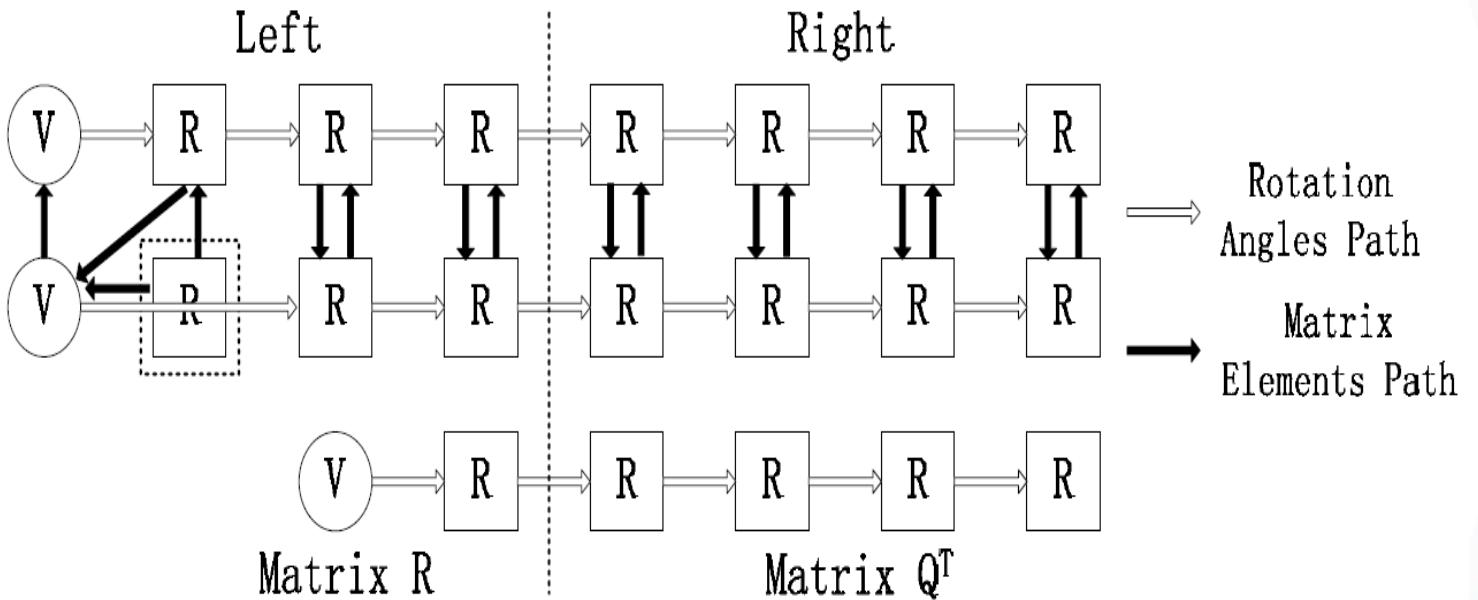
$$X_{step1} = X \cdot T_{3,1} \cdot T_{4,1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{6} & \frac{1}{9} \\ \frac{1}{3} & \frac{3}{4} & \frac{1}{7} & \frac{1}{8} \\ \frac{2}{5} & \frac{1}{4} & \frac{1}{2} & \frac{1}{6} \\ \frac{3}{10} & \frac{1}{3} & \frac{2}{7} & \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.6398 & 0.7685 & 0 \\ 0 & -0.7685 & 0.6398 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.6402 & 0 & 0 & 0.7682 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.7682 & 0 & 0 & 0.6402 \end{bmatrix} = \begin{bmatrix} 0.3905 & 0.5759 & 0.3266 & 0.2631 \\ 0.5205 & 0.6720 & 0.4757 & 0.2083 \\ 0 & -0.4164 & 0.2100 & 0.0108 \\ 0 & -0.1709 & 0.0548 & 0.0748 \end{bmatrix}$$





# A Novel Systolic Array Architecture for QR

- The second step:





# QR Decomposition (4×4)

$$X_{step1} = \begin{bmatrix} 0.3905 & 0.5759 & 0.3266 & 0.2631 \\ 0.5205 & 0.6720 & 0.4757 & 0.2083 \\ 0 & -0.4164 & 0.2100 & 0.0108 \\ 0 & -0.1709 & 0.0548 & 0.0748 \end{bmatrix}$$

**Step2**

For the first and second rows, zeroing  $x_{2,1}$

$$c_{2,1} = \frac{0.3905}{\sqrt{(0.3905)^2 + (0.5205)^2}} \approx 0.6001 \quad s_{2,1} = \frac{0.5205}{\sqrt{(0.3905)^2 + (0.5205)^2}} \approx 0.7999$$

For the third and fourth rows, zeroing  $x_{4,2}$

$$c_{4,2} = \frac{-0.4164}{\sqrt{(-0.4164)^2 + (-0.1709)^2}} \approx -0.9251 \quad s_{4,2} = \frac{-0.1709}{\sqrt{(-0.4164)^2 + (-0.1709)^2}} \approx -0.3797$$

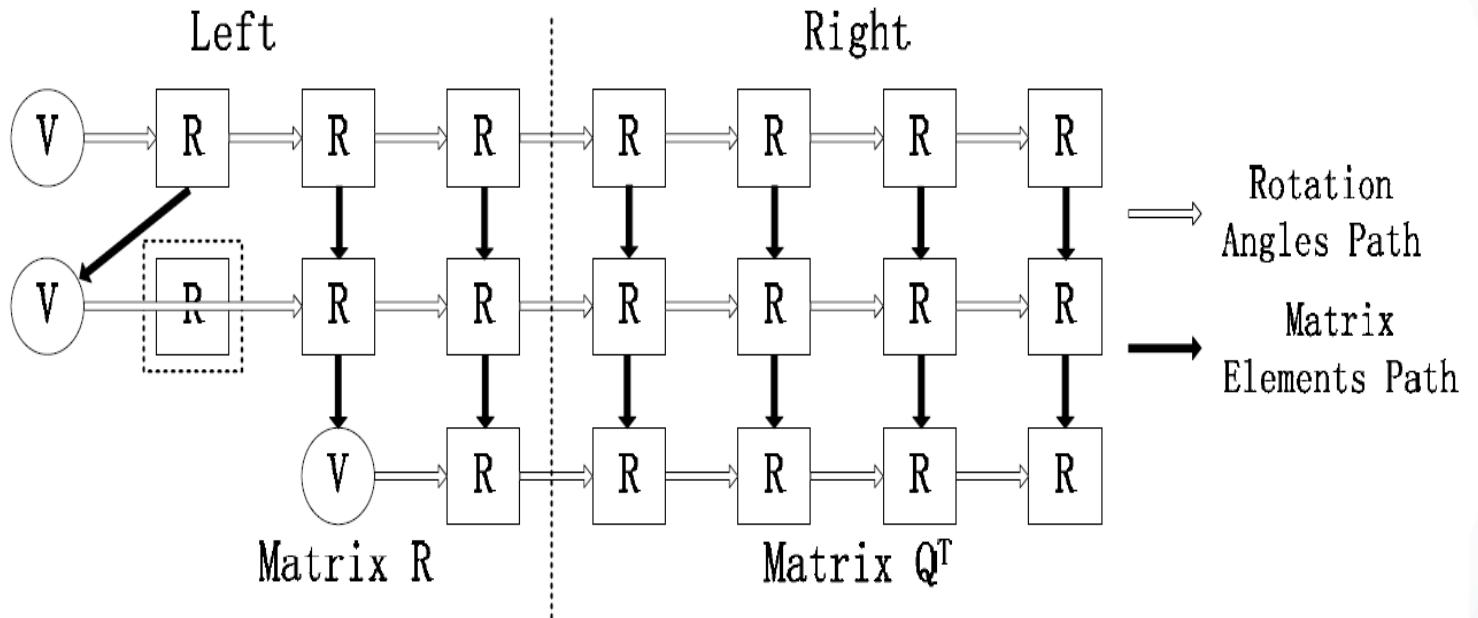
$$X_{step2} = X_{step1} \cdot T_{4,2} \cdot T_{2,1} = \begin{bmatrix} 0.3905 & 0.5759 & 0.3266 & 0.2631 \\ 0.5205 & 0.6720 & 0.4757 & 0.2083 \\ 0 & -0.4164 & 0.2100 & 0.0108 \\ 0 & -0.1709 & 0.0548 & 0.0748 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -0.9251 & -0.3797 \\ 0 & 0 & 0.3797 & -0.9251 \end{bmatrix} \cdot \begin{bmatrix} 0.6001 & 0.7999 & 0 & 0 \\ -0.7999 & 0.6001 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.6507 & 0.8831 & 0.5765 & 0.3245 \\ 0 & -0.0574 & 0.0242 & -0.0855 \\ 0 & 0.4501 & -0.2151 & -0.0384 \\ 0 & 0 & 0.0290 & -0.0651 \end{bmatrix}$$





# A Novel Systolic Array Architecture for QR

- The third step:





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# QR Decomposition (4×4)

For the second and third rows, zeroing  $x_{3,2}$

$$X_{step2} = \begin{bmatrix} 0.6507 & 0.8831 & 0.5765 & 0.3245 \\ 0 & -0.0574 & 0.0242 & -0.0855 \\ 0 & 0.4501 & -0.2151 & -0.0384 \\ 0 & 0 & 0.0290 & -0.0651 \end{bmatrix} \xrightarrow{\text{Step3}} c_{3,2} = \frac{-0.0574}{\sqrt{(-0.0574)^2 + (0.4501)^2}} \approx -0.1265 \quad s_{3,2} = \frac{0.4501}{\sqrt{(-0.0574)^2 + (0.4501)^2}} \approx 0.9920$$

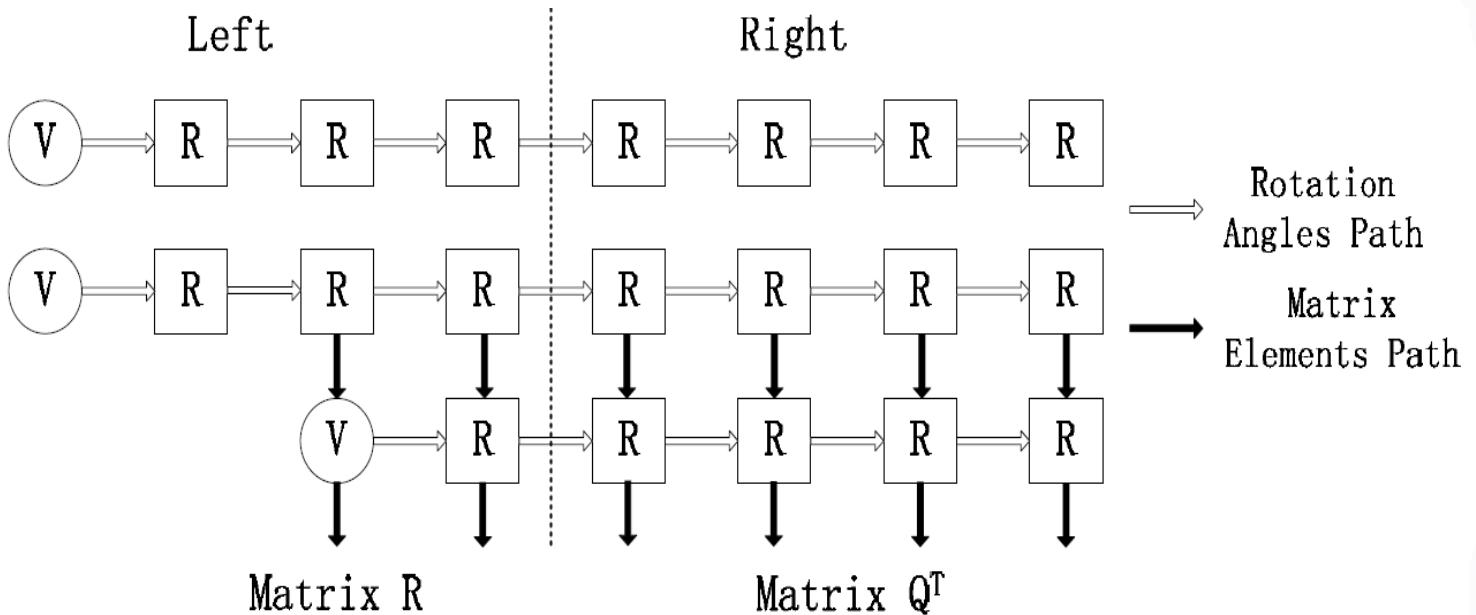
$$X_{step3} = X_{step2} \cdot T_{3,2} = \begin{bmatrix} 0.6507 & 0.8831 & 0.5765 & 0.3245 \\ 0 & -0.0574 & 0.0242 & -0.0855 \\ 0 & 0.4501 & -0.2151 & -0.0384 \\ 0 & 0 & 0.0290 & -0.0651 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.1265 & 0.9920 & 0 \\ 0 & -0.9920 & -0.1265 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.6507 & 0.8831 & 0.5765 & 0.3245 \\ 0 & 0.4538 & -0.2164 & -0.0273 \\ 0 & 0 & 0.0032 & 0.0897 \\ 0 & 0 & 0.0290 & -0.0651 \end{bmatrix}$$





# A Novel Systolic Array Architecture for QR

- The fourth step:





2

# QR Decomposition (4×4)

$$X_{step3} = \begin{bmatrix} 0.6507 & 0.8831 & 0.5765 & 0.3245 \\ 0 & 0.4538 & -0.2164 & -0.0273 \\ 0 & 0 & 0.0032 & 0.0897 \\ 0 & 0 & 0.0290 & -0.0651 \end{bmatrix} \xrightarrow{\text{Step4}} \text{For the third and fourth rows, zeroing } x_{4,3}$$
$$c_{4,3} = \frac{0.0032}{\sqrt{(0.0032)^2 + (0.0290)^2}} \approx 0.1097 \quad s_{4,3} = \frac{0.0290}{\sqrt{(0.0032)^2 + (0.0290)^2}} \approx 0.9940$$

$$X_{step4} = X_{step3} \cdot T_{4,3} = \begin{bmatrix} 0.6507 & 0.8831 & 0.5765 & 0.3245 \\ 0 & 0.4538 & -0.2164 & -0.0273 \\ 0 & 0 & 0.0032 & 0.0897 \\ 0 & 0 & 0.0290 & -0.0651 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.1097 & 0.9940 \\ 0 & 0 & -0.9940 & 0.1097 \end{bmatrix} = \begin{bmatrix} 0.6507 & 0.8831 & 0.5765 & 0.3245 \\ 0 & 0.4538 & -0.2164 & -0.0273 \\ 0 & 0 & 0.0292 & -0.0549 \\ 0 & 0 & 0 & -0.0963 \end{bmatrix} = R$$





# A Novel Systolic Array Architecture for QR

- We propose a scheme to use the sign of  $y$  not only to control the rotation direction of vectoring CORDIC, but also to control the rotation direction of the rotation CORDIC.

Original Angle $\theta$	Inputs $x_I$	Inputs $y_I$	Control Signals $[c_2 c_1 c_0]$	Transformation $\theta_T$	Outputs $x_O$	Outputs $y_O$
$[0, \frac{\pi}{4})$	$y_0$	$x_0$	000	$\theta$	$y_1$	$x_1$
$[\frac{\pi}{4}, \frac{\pi}{2})$	$x_0$	$y_0$	001	$\frac{\pi}{2} - \theta$	$y_1$	$-x_1$
$[-\frac{\pi}{2}, -\frac{\pi}{4})$	$y_0$	$x_0$	010	$\theta + \frac{\pi}{2}$	$-x_1$	$y_1$
$[-\frac{\pi}{4}, 0)$	$x_0$	$y_0$	011	$-\theta$	$x_1$	$y_1$
$[\frac{\pi}{2}, \frac{3\pi}{4})$	$y_0$	$x_0$	100	$\theta - \frac{\pi}{2}$	$x_1$	$-y_1$
$[\frac{3\pi}{4}, \pi)$	$x_0$	$y_0$	101	$\pi - \theta$	$-x_1$	$-y_1$
$[-\pi, -\frac{3\pi}{4})$	$y_0$	$x_0$	110	$\theta + \pi$	$-y_1$	$-x_1$
$[-\frac{3\pi}{4}, -\frac{\pi}{2})$	$x_0$	$y_0$	111	$-\theta - \frac{\pi}{2}$	$-y_1$	$x_1$



# Validations and Experiments

- Enhanced Vectoring CORDIC

## a) Accuracy Comparison

## b) Area

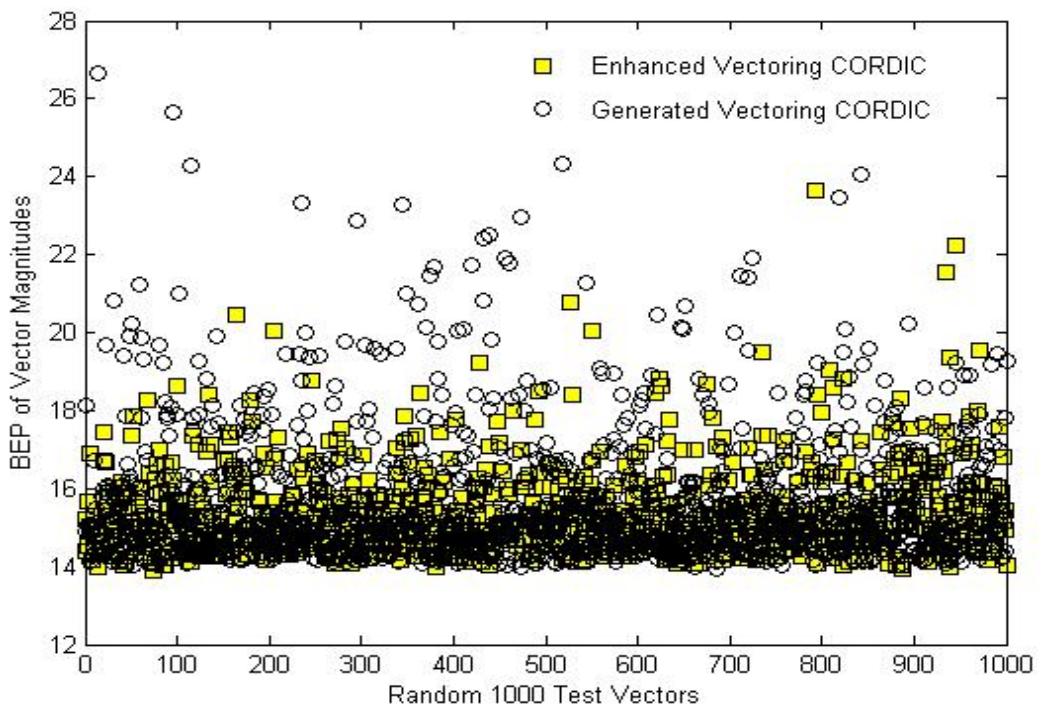
## c) Power

Comparison

Operations

Enhanced

Operations



RDIC

7.5 100

201 0.203

047 0.047

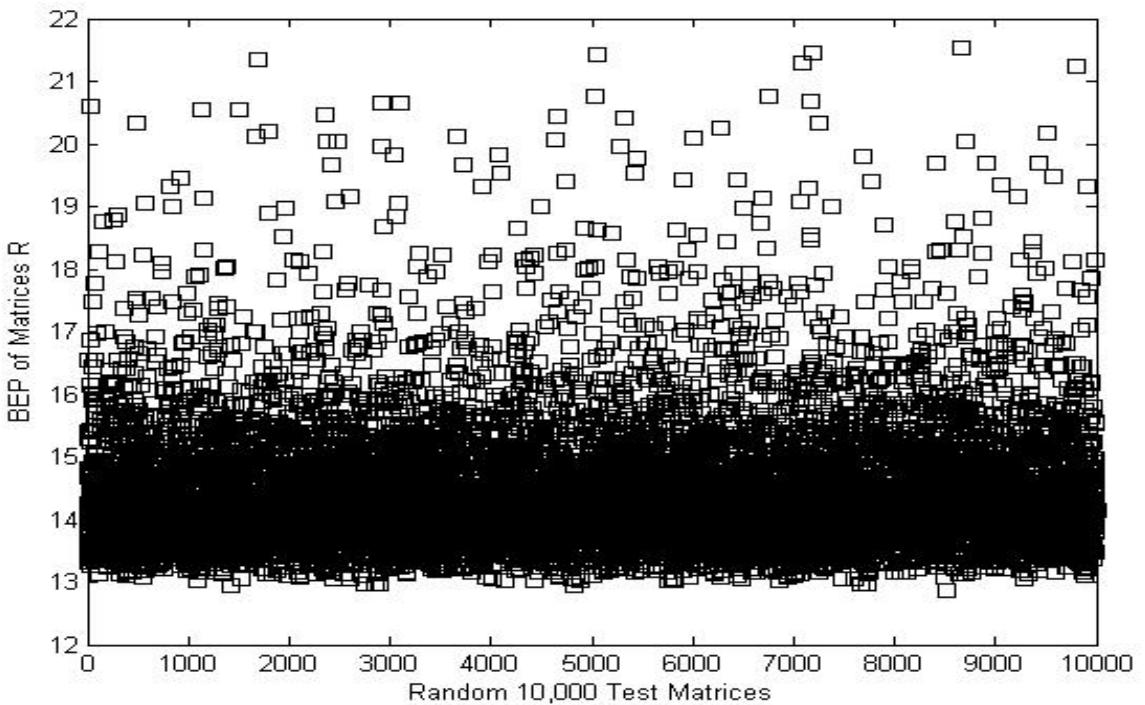


# Validations and Experiments

- Novel Systolic Array Architecture for QR

## a) Accuracy Analysis

## b) Ar



ed Version	3
22	
9122	
8.82	
54	
0.476	



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# Conclusions

- Enhanced vectoring CORDIC uses less hardware resources, provides high speed and dissipates less power
- Novel systolic array architecture for QR decomposition saves area and accelerates the process without any accuracy penalty





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# Future Work

- Apply the novel systolic array architecture for other size matrices
- Design a CORDIC chip to redo the QR and some other CORDIC based applications

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# Q and A? Thank you !

